Hyperuniformity of random graphs

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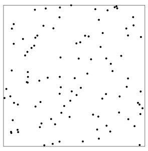
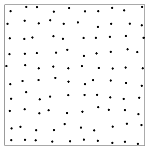
Motivation: In materials science, *hyperuniform* systems appear disordered at the microscopic level while exhibiting strongly suppressed density fluctuations over large spatial scales. This hidden order has been associated with anomalous transport, unusual mechanical behaviour, and the control of wave propagation in complex media. Recent advances have established a mature theory of hyperuniformity for point processes and random measures. Yet many physical systems are characterized not only by the locations of their constituents, but also by the network of interactions between them. Random geometric graphs encode this additional structure through bonds, contacts, or transport pathways. Extending hyperuniformity to graph-based observables could therefore provide a more faithful description of complex materials and clarify how large-scale order influences connectivity, transport, and spectral properties.

Context: A stationary point process μ is called *hyperuniform* if $\mathrm{Var}\big(\mu(B_R)\big) = o\left(|B_R|\right)$ as $R \to \infty$. That is, if its number variance grows more slowly than the volume of the observation window.



(a) Non-hyperuniform point process



(b) Hyperuniform point process

A central quantitative descriptor is the *Bartlett spectral measure*, defined as the Fourier transform of the covariance measure, or its density S, called the *structure factor*. Its behaviour near the origin determines the suppression of large-scale fluctuations.

For a random graph, however, there is no unique analogue of particle density: one may count vertices, weight them by degree, measure the number or total length of edges, or include conductances and other marks. These observables may exhibit different fluctuation regimes. The central question is therefore which graph observables provide meaningful analogues of hyperuniformity and how their low-frequency behaviour reflects the geometry and topology of the network.

Figure 2: Voronoi tessellation, Delaunay triangulation and 2-nearest-neighbour graph constructed from a non-hyperuniform input (left) and a hyperuniform input (right).

Tasks: The project will address the following questions:

1. Identify canonical stationary random measures associated with random graphs and tessellations, such as vertex-, degree-, edge-, and length-weighted measures, and define their corresponding Bartlett spectra.
2. Determine when hyperuniformity of the underlying point process is inherited by graph- and tessellation-based observables, in particular for nearest-neighbour, Delaunay, and Voronoi constructions.
3. Relate the low-frequency fluctuation behaviour of these observables to edge geometry, connectivity, and the spectra of Laplacian-type operators, and study its stability under finite-cost optimal transport.

References

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[2] C. E. Maher, J. L. Marzuola and K. A. Newhall, *Probing the influence of topological and geometric disorder on the spectrum of the differential Laplacian operator on networks*, Preprint (2026).