Report from the Focussed Innovative Session on Non-Hermitian Quantum Mechanics

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March 4, 2026

1 Introduction

On the 18th and 19th of December 2025, the above five researchers met to discuss Non-Hermitian Quantum Mechanics at Lancaster University in a Focussed Innovative Session funded by the mSPACE COST Action CA24122. They had different interests namely computational spectral theory (Drysdale and Colbrook), theoretical aspects of non-self-adjoint Quantum Mechanics (Krejčířík) and physical applications (Bender and Woodley). Individuals were invited to give informal talks from their perspectives, resulting in the discussion on the value of creating a rigorous mathematical basis for non-Hermitian quantum mechanics. In particular, a viewpoint was established that it is physically quite artificial to consider a system to be totally energetically isolated from an external environment, as in the Hermitian case, even though it is mathematically convenient. Describing particle (or more fundamentally field) interactions in non-Hermitian terms could therefore be a natural and physically fundamental way to orient some of our mathematical discussions moving forward.

The article central to discussion was the following "Computation and Verification of Spectra for Non-Hermitian Systems" [1], in which verified eigenvalues of the imaginary cubic oscillator $H_B = p^2 + ix^3$ were calculated. This operator is a primary example in non-Hermitian quantum mechanics, as it exhibits PT-symmetry and has purely real spectrum. However, no metric operator exists for this operator [10]. Unbroken PT-symmetric operators exhibit CPT-symmetry, but the C operator often has to be calculated via WKBJ symmetry, eigenvalue by eigenvalue. It is for this reason that non-Hermitian quantum mechanics can be considered a "bootstrap theory" [3]. By bootstrap theory, we mean using constraints such as positivity to solve the system. The computational framework of [1] offers an approach to calculate the C operator, creating a link between what is computationally possible and physically possible.

The reason for this meeting was to extend these ideas further, namely to a rigorous Hilbert space framework, but also to ask which operators to look at next and why. In the following report, we will outline the key topics discussed, how the meeting contributed to the action objectives and what are expected future outcomes and how to achieve them.

2 Key Topics Discussed

- How do WKBJ expansions relate to the Computational Spectral Problem?

  The computational spectral problem is the following; a primary set, Ω, that describes the input class, a metric space (M, d), a problem function $\Xi : \Omega \longrightarrow M$ and an evaluation set Λ. For the classical computational spectrum $\Omega = \Omega_{l^2(\mathbb{N})}$, $M = M_{AW}$, $\Xi = Sp(A)$ and $\Lambda = \{A \longrightarrow \langle Ae_i, e_j\rangle$ where $i, j \in \mathbb{N}\}$, i.e. we are working on the space of square summable sequences, where our outputs can be related by an Attouch-Wets metric, trying to compute the spectrum of an operator A where we have access to the entries of the discretised matrix. In [4], it was proven that without extra information this problem cannot be computed in under three limits. Under certain circumstances, the spectral problem can be computed in one limit, but this relies on having two assumptions. The first assumption is that the operator can be represented in a banded matrix. The second assumption revolves around having bounds on the resolvent norm so one can perform a local search. The WKBJ [5, 6] expansions translate into bounds on the resolvent norm thus allowing the fulfilment of the second assumption and allowing us to "lasso" the eigenvalue. In this way, we can bring together traditional eigenvalue approximation methods with modern computation techniques for verified pseudospectra.

- How do basis properties relate to CPT-symmetry?

  It is problematic that we do not know the properties of the C-operator, because as demonstrated in [7] the spectrum is not necessarily preserved under unbounded transformations. Therefore, it is important to know how the C operator relates to the standard L^2 space, which we use in computation. Fortunately, a solution arises by restricting the domain of operator A to a specific set of functions that take the metric space into consideration. This was explored in [8] with respect to the linear operator in the real non-self-adjoint Ginzburg Landau operator to allow for stochastic homogenisation of the nonlinear equation.

  Regarding the imaginary cubic oscillator an elegant solution may be found by having conditions on the domain in terms of the condition number - this will form the basis of an article following the Focussed Innovative Session. The idea is that by ensuring the C-operator is a bounded by restricting the domain, we make sure the spectrum is conserved. In this case, we may have to relax $C^2 = 1$, which is an assumption normally used to work out the C operator. However, as stated previously, non-Hermitian quantum mechanics is a "bootstrap theory" with assumptions that need to be reconsidered.

- A new problem: The Spectrum of the Complex Magnetic Harmonic Oscillator on the Disk

  In the workshop, we calculated the spectrum of the magnetic Laplacian with complex homogeneous field on the disk. This operator is given by

$$(-i\nabla - A)^2 \tag{1}$$

  where $A(x, y) = \frac{\beta}{2}(-y, x)$ where β is allowed to to be non-real. After expanding and rescaling, we have the following spectral problem

$$K = -\frac{1}{R^2}\Delta - i\beta(y\partial_x - x\partial_y) + \frac{R^2\beta^2}{4}(x^2 + y^2). \tag{2}$$

  To calculate this, we used operator folding [4].This was necessary, since spectral pollution was otherwise obtained when using general finite sections, as one can see in Figure 1. We also contrasted this to the solution to the problem solved explicitly with special functions. The red eigenvalues were calculated via the finite-section method and we can see that they do not lie in the pseudospectral contours. Additional computations done in the Focussed Innovative Session found this operator to have non-trivial essential numerical range [9], and this will form the basis of another paper.

3 How did we contribute to COST action objectives?

The Focussed Innovative Session contributed directly to the mission of the mSPACE COST Action by strengthening coordination across mathematical subfields and by consolidating a collaborative environment for new research directions in non-Hermitian quantum mechanics. The meeting brought together complementary expertise spanning computational spectral theory, theoretical operator analysis, and mathematical physics applications, with discussion explicitly focused on how transferable mathematical techniques (e.g., resolvent bounds, WKBJ asymptotics and operator folding) can be translated across modelling paradigms and embedded into rigorous Hilbert space frameworks. This aligns with the Action's research coordination objectives of leveraging collaborations across geographic areas and subdisciplines, and of advancing mathematically grounded approaches to complex systems through spectral-theoretic analysis.

In addition, the session supported the Action's capacity-building objectives through inclusive participation and early-career development. The meeting included researchers at early stages of their career, enabling structured exposure to advanced tools and open problems at the interface of theory and computation, and strengthening pathways for future collaboration through joint outputs (including work currently in progress following the session). The participant group also reflected COST's inclusion priorities: supporting the establishment of a sustainable, diverse network of researchers working across analytic, numerical, and physical aspects of spectral problems. Overall, the session contributed to the Action by deepening cross-community connectivity, broadening participation, and accelerating collaborative outputs aligned with the Action's long-term goals.

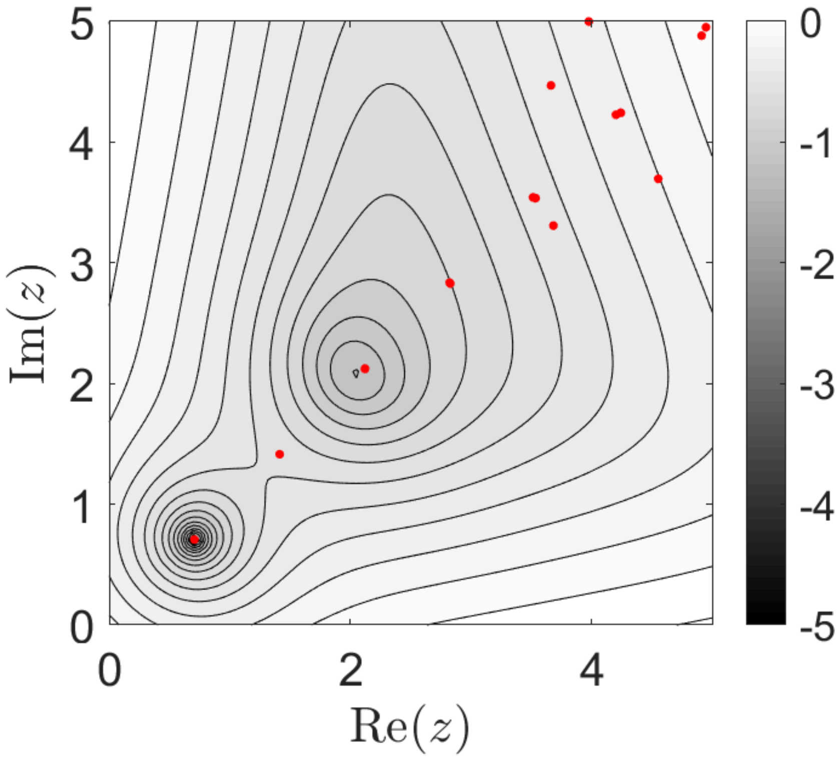

Figure 1: The eigenvalues of (2) calculated via the finite section method. The pseudospectrum is calculated via operator folding techniques and therefore tells us precisely where the spectrum should be.

4 Expected Outcomes

This meeting has already resulted in concrete follow-on activity and ongoing research collaboration. In particular, new collaborations have been established between researchers working on distinct aspects of non-self-adjoint quantum mechanics, linking complementary expertise in spectral theory, pseudospectra, semi-classical methods, and computational verification. These collaborations are now being developed through regular follow-up discussions, with participating researchers holding weekly online meetings to progress joint work and refine shared research directions. The intention is that these collaborations will lead to one or more joint research articles aimed at high-profile venues in mathematical physics, including journals such as Physical Review Letters, alongside additional papers targeting specialist journals in spectral theory and operator analysis. Additionally, there will be follow-up opportunities for the members of this Focussed Innovative Sessions to meet namely at the "International School of Multiscale Mathematical Models for Multi-Agent Systems" at the Ettore Majorana Foundation and Centre for Scientific Culture in Italy (2nd to 6th March 2026), Spectral Theory by the Lake in Lancaster, UK (9th and 10th April) also "Mathematical Aspects of the Physics with Non-self-adjoint operators" in Marseilles (20 - 24th April 2026), France.

A further outcome is strengthened cross–Working Group interaction within the COST Action. The collaborations established through the session support direct "working group pollination". This cross-WG structure helps ensure that theoretical and computational developments circulate across the Action, accelerating shared objectives and supporting future joint initiatives such as STSMs, follow-up workshops, and joint proposal activity.

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