The first meeting of Working Group 1 took place at Stockholm University on December 6-8, 2025 under the name *QGraph-2025*. This is a series of conferences organized approximately every year in direct connection to Nobel Prize lectures in Physics and Chemistry. We followed usual format: about 25-30 researchers working in the area of differential equations on metric graphs. Small format of the conference and rather focused subject allows free exchange of ideas and better understanding of presented lectures by the participants.

It is customary that all participants present short (25 minutes lectures) presenting recent achievements. This format suits our needs since most of the participants know the subject and lectures may skip unnecessary introductions into the subject. This year we started the conference with three introductory lectures aiming to show members of the group three possible directions, where our research may develop during the COST program.

These lectures were given by:

- Anna Rozanova-Pierrat
  Spectral stability under domain convergence in a class of non-Lipschitz uniform domains

- David Krejcirik
  Spectrally optimal shapes

- Colette De Coster
  Nonlinear Schrödinger equation on metric graphs: Introduction and Open Question

The remaining program contained 19 short lectures, where participants presented their recent results and discussed future problems.

At the end of the first two days special discussion sessions were arranged, where participants discussed new ideas and possible future collaborations in an informal atmosphere.

The event in addition to COST was sponsored by the Swedish Research Council (VR), Wallenberg Initiative for Networks and Quantum Information (WINQ) and the Dept. of Mathematics, Stockholm Univ.

Organizers: Alice Brolin, Matthew de Courcy-Ireland, Mina Farag, Pavel Kurasov

Programme

All lectures of the meeting will take place at Stockholm University (Cramér room), and on Zoom via

`https://stockholmuniversity.zoom.us/j/66019466760`

The schedule refers to Stockholm time (CET).

Saturday, 6 December

	Chair: Pavel Kurasov
10.00-10.50	A. Rozanova-Pierrat
10.50-11.10	**Coffee break**
11.10-12.00	D. Krejcirik
12.10-13.00	C. De Coster
13.00-14.00	**Lunch**
14.00-14.25	P. Exner
14.30-14.55	D. Mugnolo
15.00-15.25	M.-E. Pistol
15.30-15.55	S. Akduman
16.00-16.30	**Coffee break**
16.30-16.55	R. Band
17.00-17.25	M. Ławniczak
17.30-17.55	G. Annalinda Neglia
18.00-18.25	J. Rohleder
18.30-21.00	**Dinner** (at the department)

Sunday, 7 December

	Chair: Matthew de Courcy-Ireland
09.30-09.55	U. Smilansky
10.00-10.25	K. Naderi
10.30-11.00	**Coffee break**
11.00-11.25	A. Brolin
11.30-11.55	D. Cao
12.00-12.25	G. Sofer
12.30-14.00	**Lunch**
14.00-14.25	M. Kučera
14.30-14.55	J. Waclawek
15.00-15.30	**Coffee break**
15.30-15.55	J. Kennedy (ZOOM)
16.00-16.25	J. Kerner (ZOOM)
16.30-16.55	V. Pivovarchik (ZOOM)
17.00-17.25	L. Alon (ZOOM)
18.00-21.00	**Conference dinner (at the department)**

Monday, 8 December

09.00-10.55	Nobel lectures in physics in Aula Magna (Stockholm Univ.) and via `https://www.kva.se/evenemang/the-nobel-lectures-2025/`
11.20-13.10	Nobel lectures in chemistry
14.30-16.20	Nobel lectures in economics

Quantum Open Books: Structure, Self-Adjointness, and Future Directions

Setenay Akduman (Izmir Democracy University, Izmir, Türkiye)

Quantum graphs have long served as effective models for thin structures, offering a convenient analytical framework for studying complex systems in mathematical physics. In several physical contexts, however, such as photonic crystals or dynamical systems, one encounters stratified structures that naturally motivate broader modelling frameworks than those provided by quantum graphs. These settings lead to the consideration of "quantum open book structures", higher-dimensional simplicial complexes equipped with a differential operator that offer a more suitable modelling perspective. The term "open book" reflects the geometry of these structures, consisting of multiple higher-dimensional "pages" meeting transversely along a common "binding".

This talk develops the analytic framework for defining the associated differential operators and examines the main structural aspects of quantum open book structures. After giving an exact definition of such structures, the conditions for self-adjointness of the open-book operator will be presented, which reflect the key results obtained to this point.

Several directions for future research will be discussed as well, including questions concerning the analyticity of the dispersion relation — namely, how the spectrum depends on variations of the binding conditions — and the challenges associated with extending the construction to more general stratified structures.

I thank Peter Kuchment for his guidance, remarks, and comments during the preparation of this note.

Nodal Statistics from Quantum Graphs to GOE Matrices: Universality and Its Failure

Lior Alon (MIT, USA)

Ever since the early work of Uzy Smilansky and collaborators, nodal count statistics have served as a probe of quantum chaos. For quantum graphs with incommensurate lengths, Smilansky, Gnutzmann, and Weber conjectured that the nodal-count fluctuations should be Gaussian, independent of the graph's structure, in the limit of large graphs. Together with Band and Berkolaiko, we later made this conjecture precise, provided numerical evidence, and identified families of graphs for which it holds.

This viewpoint motivated the analogous question for operators on discrete graphs, where numerical experiments again suggested universal Gaussian behavior. Analytic progress required studying randomly signed graphs, due to the nodal-magnetic theorem and Band's Morse-theoretic approach. While Urschel and I disproved universality in the unsigned case, the conjecture for randomly signed graphs remained open and appeared plausible.

In this talk I will present a new result with Urschel and Mikulincer showing that the nodal count of GOE matrices converges to the semicircle law, thereby disproving the universality conjecture even for the randomly signed case.

Can one color the shape of a (Kohmoto) butterfly?

Ram Band (Technion, Haifa, Israel)

The spectra of the Kohmoto model give rise to a fractal phase diagram, known as the Kohmoto butterfly. The butterfly encapsulates the spectra of all periodic Kohmoto Hamiltonians, whose index invariants are sought after. Topological methods are ill defined due to the discontinuous periodic potentials, and hence fail to provide index invariants. We discuss how to overcome that obstacle and provide a complete classification of the Kohmoto model indices – suggesting new physical invariants instead of Chern indices. Our approach encodes the Kohmoto butterfly as a spectral tree graph, reflecting the quasiperiodic nature via the periodic spectra. This yields a complete coloring of the phase diagram and a new perspective on other spectral butterflies. The talk is based on a joint work with Siegfried Beckus.

Ambarzumian Theorem for Quantum Graphs with Magnetic Potential

Alice Brolin (Stockholm University, Sweden)

According to the Ambarzumian theorem by E.B. Davies, a Schrödinger operator with non-zero (electric) potential on a metric graph will have different spectrum than the Laplacian on the graph (assuming the operators have standard vertex conditions). We show how this theorem can be generalized to also include operators with magnetic potentials. There are operators with non-zero magnetic potential that are isospectral to the Laplacian, however we show that this can only happen if the electric potential is zero, and the magnetic potential can be eliminated by a gauge transformation in such a way that the vertex conditions are still standard.

Geodesic nets via quantum graph's eigenvalue optimisation

Duc Cao (King's College, London, UK)

In this talk, we explore a connection between geodesic nets and quantum graphs optimising certain functionals from spectral theory. For surfaces, critical metrics for the normalised k-th eigenvalue of the Laplacian give rise to isometric minimal immersions to a unit sphere. In this spirit, we obtain geodesic nets from optimal quantum graphs, and obstructions to the existence of critical metrics.

Nonlinear Schrödinger equation on metric graphs: Introduction and Open Questions.

Colette De Coster (UPHF, Valenciennes, France)

In this talk, we introduce different techniques in order to study the nonlinear Schrödinger equation on metric graphs

$$\begin{cases} -u'' + \lambda u = |u|^{p-2}u & \text{on every edge } e \in \mathbb{E} \\ u \text{ is continuous} & \text{on } \mathcal{G} \\ u(\mathrm{v}) = 0 & \text{at every vertex } \mathrm{v} \in \mathbb{V}_0 \\ \displaystyle\sum_{e \succ \mathrm{v}} u'_e(\mathrm{v}) = 0 & \text{at every vertex } \mathrm{v} \in \mathbb{V} \setminus \mathbb{V}_0 \end{cases}$$

where $p > 2$, $\lambda > 0$, $\mathcal{G} = (\mathbb{E}, \mathbb{V})$ is a metric graph.

We will try to introduce phase plane analysis techniques as well as some variational methods used to study such kind of problems and show their advantages and limitations.

If time is enough, we will also introduce the Lyapunov-Schmitt reduction method in order to have symmetry informations on the solutions.

Quantum transport in a homogeneous magnetic field.

Pavel Exner (Doppler Institute, Prague, Czech Republic)

We discuss transport in two-dimensional quantum systems exposed to a homogeneous magnetic field due to various translationally invariant perturbations. The latter come from a potential, regular or singular, added to the two-dimensional Landau Hamiltonian. In particular, we will demonstrate situations in which such a magnetic transport has no classical counterpart. If time allows, we will also mention the three dimensional case of adjacent layers, coupled by an infinitely long window in the common boundary.

On eigenvalue ratios of quantum trees

James Kennedy (University of Aveiro, Portugal)

A now-classical inequality of Ashbaugh–Benguria for the Dirichlet Laplacian on domains states that the ratio of the first two eigenvalues is maximised when the domain is a ball. On quantum graphs, an old inequality of Nicaise from the 1980s establishes that the corresponding ratio is bounded from above on quantum trees with Dirichlet conditions at all leaves ("Dirichlet trees"); however, it was shown about 15 years ago by Demirel and Harrell that, if there is at least one Dirichlet condition but Neumann leaves and cycles are allowed, then in general there is no upper bound.

We will show that an upper bound can be recovered – on all eigenvalue ratios, not just the first two – in terms of the number of Neumann leaves and cycles. We will also show that for many Dirichlet trees a sharp Ashbaugh–Benguria-type bound holds, with the maximising graphs being equilateral stars on any number of edges (this includes the interval).

This is based on ongoing joint work with Evans M. Harrell II (Georgia Tech) and Gabriel Ramos (Lisbon).

Atypical spectral and transport properties of non-locally finite crystals (and maybe more)

Joachim Kerner (University of Hagen, Germany)

In the first part of the talk we discuss recent results on Schrödinger operators on periodic graphs which are non-standard in the sense that we allow vertices to have an infinite number of neighbours. It turns out that such non-locally finite graphs exhibit various phenomena which are absent in the locally finite setting: and this is true from a spectral as well as transport point of view. Using some explicit examples, we shall illustrate such new effects in more detail. Quite surprisingly, it turns out that one of the examples provides us with a negative answer to a question raised by Damanik et al. in a recent paper on ballistic transport (this part of talk is based on joint work with O. Post, M. Sabri, and M. Täufer). If time allows, we shall also quickly discuss spectral comparison results on discrete graphs. In recent years, various authors have derived such comparison results on Euclidean domains and quantum graphs. Our aim is to present a generalization to the discrete setting. Along the way, we also establish a so-called local Weyl law which is of independent interest (the second part of the talk is based on joint work with P. Bifulco and C. Rose)

Spectrally optimal shapes.

David Krejcirik (Czech Technical University, Prague, Czech Republic)

Survey of old questions, fresh progresses and new challenges.

Spectrum of the wave equation with Dirac damping on a compact star graph.

Mikuláš Kučera (Czech Technical University in Prague, Czech Republic)

We consider the wave equation with a distributional Dirac damping and Dirich- let boundary conditions on a compact interval. It is shown that the spectrum of the corresponding wave operator is fully determined by zeroes of an entire function. Consequently, a considerable change of spectral properties is shown for certain critical values of the damping parameter. We also derive a definitive criterion for the Riesz basis property of the root vectors for an arbitrary place- ment of a complex-valued Dirac damping. Finally, we consider a generalisation of the problem for compact star graphs and provide insight into the essence of the critical damping constant.

Isoscattering without Isospectrality in Quantum Graphs: Experiments with Microwave Networks

Michal Ławniczak (Institute of Physics PAS, Warsaw, Poland)

We will present an experimental demonstration of the isoscattering phenomenon in microwave networks that simulate quantum graphs. We use the concept of germ graphs and the Titchmarsh–Weyl M-function formalism to create large families of isospectral and isoscattering graphs. This approach is completely different from the original method developed by Sunada, in which isospectral graphs are obtained as quotients of a large symmetric graph. We will also demonstrate how the M-function can predict pairs of graphs that exhibit identical scattering behaviour despite having different eigenvalue spectra.

Non-Markovian heat flows on directed hypergraphs

Delio Mugnolo (FernUniversität Hagen, Germany)

Recent years have seen an increased interest in the analysis of structures generalizing graphs to allow for higher-order interactions in real-life models: non-binary relations between agents are often encoded using the formalism of hypergraphs. These discrete structures share some obvious similarities and many fundamental differences with graphs: their Laplacian is generally not associated with a Dirichlet form, to mention only one serious technical difficulty. In my talk I will offer an overview of similarities and differences between gradient flows on graphs and hypergraphs.

Conformal eigenvalues on metric graphs.

Kiyan Naderi (Universität Innsbruck)

On a metric graph G every measure μ gives rise to a corresponding first-order Sobolev space $H^1(G, \mu)$, with the energy form as its inner product, and a self-adjoint Laplace operator H_μ in $L^2(G, \mu)$ generated by the energy.

We investigate lower bounds for the k-th eigenvalue $\lambda_k(H_\mu)$ of all these Laplacians. More precisely, we define the so-called k-th conformal eigenvalue as the infimum of all k-th eigenvalues of Laplace operators generated by Borel measures. First properties of these new graph invariants are investigated: On the one hand, the first conformal eigenvalue is closely related to the resistance metric for graphs, random walks, and spectral graph partitioning. On the other hand, we have studied the asymptotic behavior of the higher conformal eigenvalues and proved a Weyl law.

Territories as multiscale systems.

Giulia Annalinda Neglia (Polytechnic University of Bari, Department of Architecture, Construction and Design, Italy)

Territories are multiscale systems where intricate dynamics arise from interactions across various spatial and temporal scales. Saverio Muratori's notion of territory as a multiscale system shaped by spatial, social and historical relations enables us to consider territory as a living organism, as well as providing a rigorous basis for understanding and analysing multi-scalar territorial, urban and architectural connections. In this framework, the interaction between elements, structures, systems and organisms explains landscape forms. Typomorphological analysis provides a powerful interpretive framework for understanding complex systems, enabling us to draw conclusions about the variety of spatial forms (architectural, urban and territorial), with the aim of advancing our understanding of the behaviour of natural and human-made systems at different levels of complexity.

Isospectral quantum graphs and eigenvalue zero

Mats-Erik Pistol (Lund, Sweden)

When studying the isospectrality of quantum graphs it is necessary to treat eigenvalue zero correctly. Standard methods using the secular determinant do not work well. For standard boundary conditions this is seldom a problem, but for more general boundary conditions care has to be taken. We have investigated the zero eigenvalue for all permutation-invariant boundary conditions for quantum graphs. We will give a set of conditions that the boundary conditions need to satisfy in order for eigenvalue zero to exist. We will also demonstrate computer generated eigenfunctions corresponding to eigenvalue zero and give examples of quantum graphs which are isospectral, but where the boundary conditions are different for the two graphs.

Spectral Flow and Eigenvalue Comparison for Schrödinger Operators on Metric Graphs

Gilad Sofer (Technion, Haifa, Israel)

When comparing the spectra of two self-adjoint operators, it is often useful to com- pare their eigenvalue counting functions. A well-known result in this direction for Schrödinger operators on metric graphs is known as Dirichlet-Neumann bracketing, which essentially states that the eigenvalue counting functions of the Dirichlet and Neumann Laplacians differ by at most the size of the boundary. The goal of this talk is to introduce a useful tool for such comparison results, known as the spectral flow, which is a topological invariant associated with one-parameter families of self-adjoint operators. We show that for Schrödinger operators on metric graphs, the spectral flow can be effectively computed using the associated scattering matrices. We then present several applications, in the form of nodal index theorems and eigenvalue interlacing results. The talk is based on joint work with Ram Band and Marina Prokhorova.

Sturm-Liouville problems on graphs with Robin boundary conditions.

Vyachaslav Pivovarchik (South Ukrainian National Pedagogical University, Odessa, Ukraine)

We study characteristic functions and describe asymptotics of the eigenvalues for the spectral Sturm-Liouville problem on graphs equipped with Robin-Kirhhoff boundary conditions. Also, we show how to recover the coefficients in the Robin conditions for the quantum graphs provided the shape of the graphs and some Robin eigenvalues are known.

Quantum trees which maximize higher eigenvalues are unbalanced

Jonathan Rohleder (Stockholm University, Sweden)

The isoperimetric problem of maximizing all eigenvalues of the Laplacian on a metric tree graph within the class of trees of a prescribed average edge length is studied. While it has been known from earlier work that the lowest positive eigenvalue is maximized by equilateral star graphs, we show that the trees maximizing the higher eigenvalues are less balanced in their shape - an observation which is also known from numerical results on the optimization of higher eigenvalues of Laplacians on Euclidean domains.

Spectral stability under domain convergence in a class of non-Lipschitz uniform domains

Anna Rozanova-Pierrat (CentraleSupélec, Univ. Paris-Saclay, France)

I will present recent results from **[1]** and **[2]** (see also **[3,4]** for the independant on the boundary measure trace theory), focusing on the stability questions under domain convergence. Firstly, I will introduce the functional framework of the trace operator allowing to work with boundaries as supports of the upper regular Borel measures **[4]**. Hence, these supports can define non-Lipschitz, and possibly fractal/multi-fractal, boundaries. In this framework, I introduce generalized Dirichlet, Neumann, and Robin problems for Poisson-type equations, for which we proved the Mosco convergence of the associated energy functionals along sequences of suitably converging domains. Generally, the Mosco convergence does not imply the operator norm convergence of resolvents. I will present the sufficient conditions on the domain convergence which imply a stability result for weak solutions, the norm convergence of the associated resolvents, and the convergence of the corresponding eigenvalues and eigenfunctions, which I will illustrate by the numerical results. Further investigations of analogous questions for the infinite graphs passing by Dirichlet forms could be a natural development.

In the end of my talk, if I have some time, I will finish with the existence of optimal shapes for the Robin boundary problems (not known before **[1]**) in the parametrized classes of admissible domains in the sense that they minimize the initially given energy functionals. The keys for this result are the uniform on the shape of the domains the Poincaré inequality and the compactness of the introduced parametrized classes of admissible domains. An alternative shape optimization method for a partially absorbing fixed boundary can be found in **[5]**, which highlights the improved absorption properties of partially absorbing shapes compared to fully absorbing ones.

References

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[2] M. Hinz, A. Rozanova-Pierrat, A. Teplyaev, Non-Lipschitz uniform domain shape optimization in linear acoustics. *SIAM J. Control Optim.* **59**,

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[3] G. Claret, M. Hinz, A. Rozanova-Pierrat, A. Teplyaev, Layer potential operators for transmission problems on extension domains. Preprint, 2024 `https://hal.science/hal-04505158`
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Pseudo Hermitian matrices in spectral graph theory (metric, discrete, weighted graph).

Uzy Smilansky (Weizmann Institute of Science)

A Pseudo Hermitian matrix is a real asymmetric matrix which is related to its Hermitian conjugate by multiplying from left and right by a regular matrix and its inverse. The classical analogues of the unitary evolution operator in e.g., quantum graphs, belong to this class – a fact which to the best of my knowledge was not studied till now. I shall discuss their spectra and in particular the spectral transition from purely real spectrum to a mixed real and (pairs of conjugate) complex eigenvalues .

(In progress with Arkady Kurnosov and Sven Gnutzmann).

Optimal discrete Hardy–Rellich–Birman inequalities.

Jakub Waclawek (Czech Technical University, Prague)

We prove sufficient conditions on a parameter sequence to determine optimal weights in inequalities for an integer power ℓ of the discrete Laplacian on the half-line. By a concrete choice of the parameter sequence, we obtain explicit optimal discrete Rellich ($\ell = 2$) and Birman ($\ell \geq 3$) weights. For $\ell = 1$, we rediscover the optimal Hardy weight of Keller–Pinchover–Pogorzelski. For $\ell = 2$, we improve upon the best known Rellich weights due to Gerhat–Krejčiřík–Štampach and Huang–Ye. For $\ell \geq 3$, our main result proves a conjecture by Gerhat–Krejčiřík–Štampach and improves the discrete analogue of the classical Birman weight due to Huang–Ye to the optimal. The talk is based on a joint work with F. Štampach