Spectral theory of two-dimensional Laplacians with oblique Robin boundary conditions

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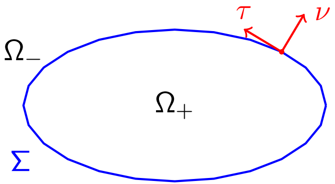
with Markus Holzmann and Georg Stenzel

Multiscale Stochastics, Patterns, and Analysis of Combinatorial Environments, March 16-19

Oblique Robin boundary conditions (ORBC)



- Σ simple, closed C^∞-curve
- $\Omega_+ \subseteq \mathbb{R}^2$ is bdd. and $\partial\Omega_+ = \Sigma$
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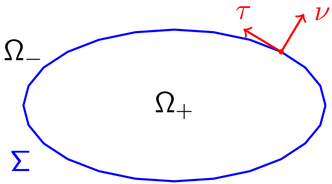

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For $\alpha \geq 0$ and $\beta \in \mathbb{R}$ study Laplacian

$$\mathcal{A}^{\pm}_{\alpha,\beta} f = -\Delta f \text{ on } \Omega_\pm \quad \text{and} \quad \partial_\nu f + i\alpha\partial_\tau f \pm \beta f = 0 \text{ on } \Sigma.$$

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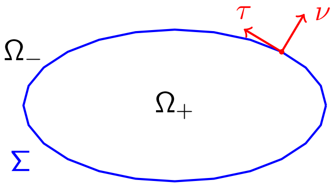

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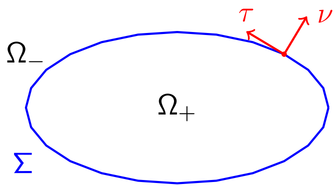

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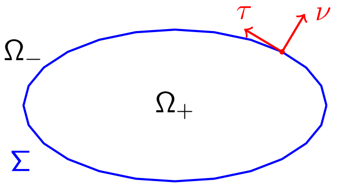

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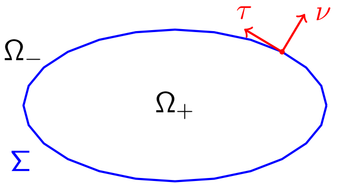

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Many contributors, e.g.: Antunes, Arendt, Arlinskii, Bade, Beals, Birman, Bögli, Bruneau, Bucur, Chill, Cossetti, Daners, Derkach, Exner, Filinov, Freeman, Freitas, Gesztesy, Giacomini, Gittins, Goffeng, Grubb, Helffer, Holzmann, Kachmar, Kennedy, Khrabustoskyi, Kovařík, Krejčiřík, Kunze, Lang, Langer, Levitin, Lions, Lotoreichik, Lou, Magenes, Malamud, Marletta, Mine, Mitrea, Mugnolo, Nazarov, Nichols, Nittka, Ouhabaz, Pankrashkin, Persson Sundquist, Plum, Popoff, Raymond, Rozenblum, Schechter, Schlosser, Schlumpf, Siegl, Taskinen, Visik, Warma, Zhu ...

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Ref: $\beta > 0$ [AntunesBenguriaLotoreichikOurmières-Benafos'21]

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$$\widehat{A}^{\pm}_{1,0} f = -\Delta f = -4\partial_z \partial_{\bar{z}} f,$$

$$\mathrm{dom}\big(\widehat{A}^{\pm}_{1,0}\big) = \big\{ f \in L^2(\Omega_\pm) \,\big|\, \partial_{\bar{z}} f \in H^1_0(\Omega_\pm) \big\}$$

Theorem

For $\alpha = 1$ and $\beta = 0$ we have:

- $\widehat{A}^{\pm}_{1,0}$ is self-adjoint in $L^2(\Omega_\pm)$ and $A^{\pm}_{1,0} \subset \widetilde{A}^{\pm}_{1,0} \subset \widehat{A}^{\pm}_{1,0}$
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Ref: [Schmidt'95] in the context of supersymmetric Dirac operators

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Main result of this talk and key takeaway:

The oblique Robin Laplacian

$$\mathcal{A}^{\pm}_{\alpha,\beta} f = -\Delta f \text{ on } \Omega_\pm \quad \text{and} \quad \partial_\nu f + i\alpha\partial_\tau f \pm \beta f = 0 \text{ on } \Sigma$$

is unbounded from below if and only if

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and in both cases the discrete eigenvalues accumulate to $-\infty$.

Some related references

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A remark on boundary value problems for the Dirac operator.
Quart. J. Math. Oxford Ser. 46 (1995), 509–516.

Thank you for your attention