The Born-Oppenheimer approximation for a 1D 2+1 particle system with zero-range interactions

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A Three-Body Problem

- 2 heavy particles (both bosons or fermions) of mass M
  - With coordinates $x_1, x_2 \in \mathbb{R}$.
- 1 light particle of mass m
  - With coordinate $x_3 \in \mathbb{R}$.

Consider the formal Hamiltonian

$$\boxed{H_{2+1}^{\mathrm{b/f}} := -\frac{1}{2M}\,\partial_{x_1}^2 - \frac{1}{2M}\,\partial_{x_2}^2 - \frac{1}{2m}\,\partial_{x_3}^2 + \beta\delta(x_3 - x_1) + \beta\delta(x_3 - x_2)}$$

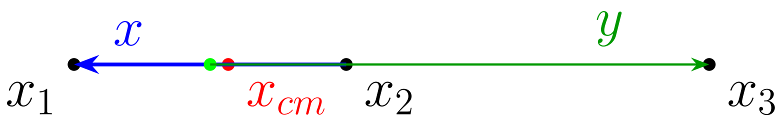
in the Hilbert space

$$L^2_{\mathrm{b/f}}(\mathbb{R}^3) := \{\Psi \in L^2(\mathbb{R}^3) : \ \Psi(x_1, x_2, x_3) = \sigma_{\mathrm{b/f}}\,\Psi(x_2, x_1, x_3)\},$$

where $\sigma_{\mathrm{b}} := +$ and $\sigma_{\mathrm{f}} := -$.

Jacobi coordinates

$$x_{cm} = \frac{M(x_1 + x_2) + m x_3}{M_{tot}}$$



$$x = x_1 - x_2 \,, \qquad y = x_3 - \frac{x_1 + x_2}{2}$$

$$M_{tot} = 2M + m \,, \qquad \mu = \frac{2Mm}{M_{tot}}$$

After factoring out the center of mass and the reduced mass we are left with

$$\boxed{H_\varepsilon^{\mathrm{b/f}} = -\varepsilon^2 \partial_x^2 - \partial_y^2 + \alpha\delta(y - x/2) + \alpha\delta(y + x/2) \qquad \text{with } \varepsilon^2 = \frac{2\mu}{M} \ll 1}$$

in the Hilbert space

$$L^2_{\mathrm{b/f}}(\mathbb{R}^2) := \{\psi \in L^2(\mathbb{R}^2) : \; \psi(x,y) = \sigma_{\mathrm{b/f}}\, \psi(-x,y)\}$$

- $\alpha \in \mathbb{R}$ independent on ε.

Leaky Quantum Graphs

Set aside the symmetry constraint, the Hamiltonian $H_\varepsilon^{\mathrm{b/f}}$ can be understood as the operator

$$-\varepsilon^2\partial_x^2 - \partial_y^2 + \alpha\delta_\Gamma \qquad \text{in } L^2(\mathbb{R}^2)$$

where $\alpha\delta_\Gamma$ is the Dirac-delta distribution supported on the star-graph with four edges

$$\Gamma = \{y = x/2\} \cup \{y = -x/2\} \subset \mathbb{R}^2$$

By the unitary transformation

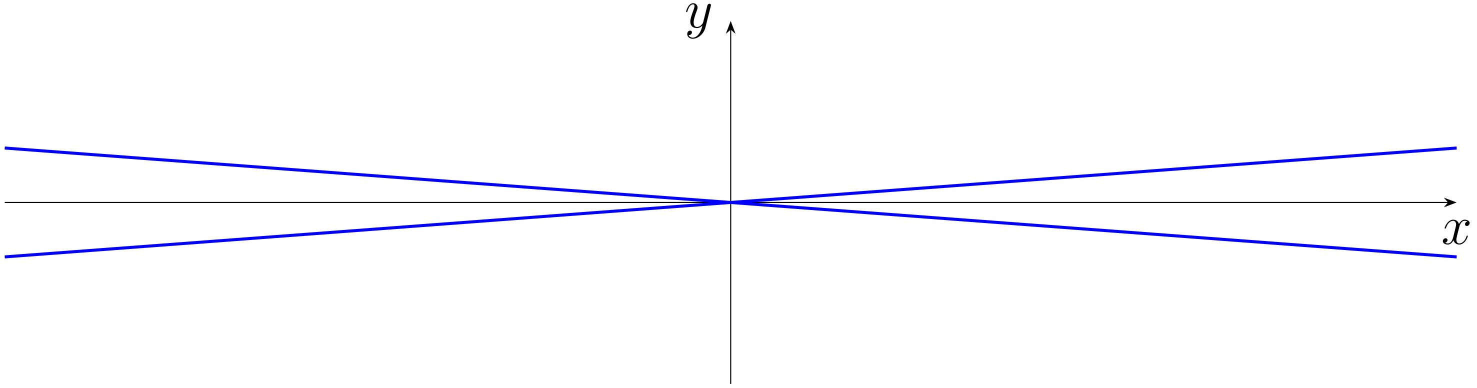
$$(U^\varepsilon\psi)(x,y) = \varepsilon^{1/2}\psi(\varepsilon x, y) \qquad (U^{\varepsilon-1}\psi)(x,y) = \varepsilon^{-1/2}\psi^\varepsilon(x/\varepsilon, y)$$

the operator above is equivalent to

$$-\Delta + \alpha\delta_{\Gamma^\varepsilon} \qquad \text{in } L^2(\mathbb{R}^2)$$

with

$$\Gamma^\varepsilon = \{y = \varepsilon x/2\} \cup \{y = -\varepsilon x/2\} \subset \mathbb{R}^2$$



Models of the form

$$-\Delta + \alpha\delta_{\mathcal{G}} \qquad \text{in } L^2(\mathbb{R}^2)$$

where $\mathcal{G}$ is a planar graph in $\mathbb{R}^2$, are called Leaky Quantum Graphs[1].

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[1] Pavel Exner, Leaky Quantum Graphs: A Review

Main Result

- Quadratic Form

$$\mathcal{B}_\varepsilon^{\mathrm{b/f}} : H^1(\mathbb{R}^2) \cap L^2_{\mathrm{b/f}}(\mathbb{R}^2) \to \mathbb{R}$$

$$\mathcal{B}_\varepsilon^{\mathrm{b/f}}(\psi) = \varepsilon^2 \|\partial_x \psi\|^2_{L^2(\mathbb{R}^2)} + \|\partial_y \psi\|^2_{L^2(\mathbb{R}^2)} + \underbrace{\alpha \|\psi|_{y=x/2}\|^2_{L^2(\mathbb{R})} + \alpha \|\psi|_{y=-x/2}\|^2_{L^2(\mathbb{R})}}_{2\alpha \|\psi|_{y=x/2}\|^2_{L^2(\mathbb{R})}}$$

- Operator

$$D(H_\varepsilon^{\mathrm{b/f}}) = \Big\{\phi \in H_{\mathrm{b/f}}^{3/2-}(\mathbb{R}^2) : \phi \in H^2(\mathbb{R}^2 \backslash (\{y = x/2\} \cup \{y = -x/2\}))$$
$$[\partial_n \phi]_{y=x/2} = \alpha \phi|_{y=x/2}$$
$$[\partial_n \phi]_{y=-x/2} = \alpha \phi|_{y=-x/2}\Big\}$$

where $[\partial_n \phi]_{y=x/2}$ denotes the jump across $\{y = x/2\}$ of the normal derivative relative to $\varepsilon^2 \partial_x^2 + \partial_y^2$; explicitely

$$\partial_n \phi = -\frac{\varepsilon^2}{2} \partial_x \phi + \partial_y \phi$$

$$H_\varepsilon^{\mathrm{b/f}} \phi = \left(-\varepsilon^2 \partial_x^2 - \partial_y^2\right) \phi \qquad (x, y) \notin \{y = x/2\} \cup \{y = -x/2\})$$

Theorem. *Let* $\alpha \in \mathbb{R}$ *and* $\varepsilon > 0$. *Then,*

$$\sigma(H_\varepsilon^{\mathrm{b/f}}) = \sigma_{ess}(H_\varepsilon^{\mathrm{b/f}}) = [0, +\infty) \qquad \textit{if} \quad \alpha \geq 0,$$

$$\sigma(H_\varepsilon^{\mathrm{b/f}}) \subseteq [-\alpha^2, +\infty), \qquad \sigma_{ess}(H_\varepsilon^{\mathrm{b/f}}) = \left[-\frac{\alpha^2}{4+\varepsilon^2}, +\infty\right) \qquad \textit{if} \quad \alpha < 0.$$

If $\alpha < 0$, *for any fixed integer* $n \geq 0$ *there exists* $\varepsilon_0 > 0$ *such that for all* $0 < \varepsilon < \varepsilon_0$ *the Hamiltonian* $H_\varepsilon^{\mathrm{b/f}}$ *has at least* $(n+1)$ *simple isolated eigenvalues*

$$-\alpha^2 < E_{\varepsilon,0}^{\mathrm{b/f}} < E_{\varepsilon,1}^{\mathrm{b/f}} < \cdots < E_{\varepsilon,n}^{\mathrm{b/f}} < -\frac{\alpha^2}{4+\varepsilon^2},$$

such that

$$E_{\varepsilon,k}^{\mathrm{b/f}} = -\alpha^2 + s_k^{\mathrm{b/f}} \alpha^2 \varepsilon^{2/3} + O(\varepsilon), \qquad \textit{for all } k = 0, \ldots, n,$$

where $s_k^{\mathrm{b}} = -\sigma_{2k}$ *and* $s_k^{\mathrm{f}} = -\sigma_{2k+1}$; *with* σ_k *interlacing negative numbers*

$$\cdots < \sigma_{2k+1} < \sigma_{2k} < \sigma_{2k-1} < \cdots < \sigma_2 < \sigma_1 < \sigma_0 < 0,$$

given by the extrema or the zeros of the Airy function Ai, *i.e.,* $\mathrm{Ai}'(\sigma_{2k}) = 0$ *or* $\mathrm{Ai}(\sigma_{2k+1}) = 0$.

Some related works.

- H. Akbas and O. T. Turgut. JMP 2018. (BO Approximation with δ-interaction)
- V. Duchêne and N. Raymond. JPA 2014. (Leaky QG - broken line)
- P. Exner and T. Ichinose. JPA 2001. (Leaky QG - asymptotically straight curve)
- K. Pankrashkin and M. Vogel. JPA 2022. (δ'-interaction on star-graph)
- P. Exner. Leaky quantum graphs: A review 2008.

From now on we focus on the case $\alpha < 0$, which is the relevant one in view of the BO approximation.

Factorization

- a fast dynamics, relative to the light particle (variable y)
- a slow dynamics, relative to the heavy particles subsystem (variable x)

Fix the relative position $x \in \mathbb{R}$ of the heavy particles.

Consider the (formal) light particle Hamiltonian h_x

$$\boxed{h_x = -\frac{d^2}{dy^2} + \alpha\delta_{x/2} + \alpha\delta_{-x/2} \qquad \text{in } L^2(\mathbb{R}, dy).}$$

Ideally:

$$H_\varepsilon^{\mathrm{b/f}} = -\varepsilon^2\partial_x^2 + \int^{\oplus} h_x dx$$

but this identity cannot be established rigorously.

- Domain of h_x

$$D(h_x) = \big\{ u \in H^1(\mathbb{R}) : u \in H^2(\mathbb{R} \backslash (\{x/2\} \cup \{-x/2\}))$$
$$[u']_{y=x/2} = \alpha u(x/2)$$
$$[u']_{y=-x/2} = \alpha u(-x/2) \big\}$$
$$h_x u = -u'' \qquad y \neq x/2, -x/2$$

- Let $\chi \in H^1(\mathbb{R}) \cup H^2(\mathbb{R} \backslash (\{0\}))$ be such that

$$\chi'(0^+) - \chi'(0^-) = \alpha \chi(0)$$

- Take a function $u_x(y) = \chi(y - x/2)$ in a neighborhood of $y = x/2$ and zero outside.
- u satisfies the boundary condition in $y = x/2$ required to belong to $D(h_x)$.
- Now take $\rho \in C_c^\infty(\mathbb{R})$ and consider $\phi(x,y) = \rho(x)\chi(y - x/2)$. This does **not** satisfy the condition

$$[\partial_n \phi]_{y=x/2} = \alpha \phi|_{y=x/2}$$

with $\partial_n \phi = -\frac{\varepsilon^2}{2}\partial_x \phi + \partial_y \phi$.

Quadratic Forms

For any fixed $x \in \mathbb{R}$

$$b_x : H^1(\mathbb{R}) \to \mathbb{R}$$

$$b_x(u) = \int_{\mathbb{R}} |u'(y)|^2 \, \mathrm{d}y + \alpha |u(x/2)|^2 + \alpha |u(-x/2)|^2$$

is the quadratic form of h_x.

Given any function $\phi \in H^1(\mathbb{R}^2)$, for a.e. $x \in \mathbb{R}$, its x-section $\phi_x(y) := \phi(x, y)$ belongs to $H^1(\mathbb{R})$. There holds

$$\mathcal{B}_\varepsilon^{\mathrm{b/f}}(\phi) = \int_{\mathbb{R}^2} \varepsilon^2 |\partial_x \phi(x,y)|^2 \mathrm{d}\mathbf{x} + \int_{\mathbb{R}} b_x(\phi_x) \mathrm{d}x \qquad \forall \phi \in H^1_{\mathrm{b/f}}(\mathbb{R}^2).$$

- ▶ b_x is bounded from below by $-\alpha^2$, hence so is $\mathcal{B}_\varepsilon^{\mathrm{b/f}}$, and $\sigma(H_\varepsilon^{\mathrm{b/f}}) \subset [-\alpha^2, +\infty)$.
- ▶ h_x has non empty discrete spectrum.
  - (E_x^{BO}, ψ_x^{BO}): ground state energy and ground state ($\|\psi_x^{BO}\| = 1$).

$$E_x^{BO} = -\lambda_0(x)$$

with

$$\lambda_0 : \mathbb{R} \to (0, +\infty)\,, \qquad \lambda_0(x) = \left(\frac{W\left(\frac{|\alpha||x|}{2}\, e^{-\frac{|\alpha||x|}{2}}\right)}{|x|} + \frac{|\alpha|}{2} \right)^2$$

Here W is the Lambert W-function:

$$W(x)e^{W(x)} = x$$

The normalized eigenfunction is

$$\psi_x^{BO}(y) := N(x) \left(e^{-\sqrt{\lambda_0(x)}\,|x/2-y|} + e^{-\sqrt{\lambda_0(x)}\,|x/2+y|} \right)$$

where $N(x)$ is the normalization constant

$$N(x) := \left(\frac{\sqrt{\lambda_0(x)}}{2\left(1 + e^{-\sqrt{\lambda_0(x)}\,|x|}\left(1 + \sqrt{\lambda_0(x)}\,|x|\right)\right)} \right)^{1/2}$$

Contribution of the GS

- Projection on the eigenfunction ψ_x^{BO}.
- This procedure produces an effective Hamiltonian for the heavy particles subsystem.
- We follow the general approach developed in D. Krejčiřík, N. Raymond, J. Royer, and P. Siegl. Mathematatika 2018.

ψ_x^{BO} is regarded as a function of two variables: $\psi^{BO}(x,y) \equiv \psi_x^{BO}(y)$.

Introduce the orthogonal projection

$$\boxed{\mathcal{P} : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)\,, \qquad \mathcal{P}\phi(x,y) := \psi^{BO}(x,y)\,\underbrace{(\psi^{BO}(x,\cdot),\phi(x,\cdot))_{L^2(\mathbb{R},dy)}}_{f_\phi(x)}}$$

$\mathcal{P}$ leaves invariant both $L^2_{\mathrm{b}}(\mathbb{R}^2)$ and $L^2_{\mathrm{f}}(\mathbb{R}^2)$; we set $\mathcal{P}^\perp := 1 - \mathcal{P}$.

The quadratic form

$$D(\widehat{\mathcal{B}}_\varepsilon^{\mathrm{b/f}}) := H^1_{\mathrm{b/f}}(\mathbb{R}^2) \qquad \widehat{\mathcal{B}}_\varepsilon^{\mathrm{b/f}}(\phi) := \mathcal{B}_\varepsilon^{\mathrm{b/f}}(\mathcal{P}\phi) + \mathcal{B}_\varepsilon^{\mathrm{b/f}}(\mathcal{P}^\perp\phi)$$

is well-defined, closed and bounded from below and it defines a self-adjoint operator $\widehat{H}_\varepsilon^{\mathrm{b/f}}$:

$$\boxed{\widehat{H}_\varepsilon^{\mathrm{b/f}} = \widehat{H}_{\varepsilon,\mathcal{P}}^{\mathrm{b/f}} \oplus \widehat{H}_{\varepsilon,\mathcal{P}^\perp}^{\mathrm{b/f}}.}$$

- $\widehat{H}_{\varepsilon,\mathcal{P}}^{\mathrm{b/f}}$ is an operator in $\mathrm{ran}(\mathcal{P}|L^2_{\mathrm{b/f}}(\mathbb{R}^2))$
- $\widehat{H}_{\varepsilon,\mathcal{P}^\perp}^{\mathrm{b/f}}$ is an operator in $\mathrm{ran}(\mathcal{P}^\perp|L^2_{\mathrm{b/f}}(\mathbb{R}^2))$
- $\sigma(\widehat{H}_\varepsilon^{\mathrm{b/f}}) = \sigma(\widehat{H}_{\varepsilon,\mathcal{P}}^{\mathrm{b/f}}) \cup \sigma(\widehat{H}_{\varepsilon,\mathcal{P}^\perp}^{\mathrm{b/f}})$
- $\sigma(\widehat{H}_{\varepsilon,\mathcal{P}}^{\mathrm{b/f}}) \subseteq [-\alpha^2, +\infty)\,, \qquad \sigma(\widehat{H}_{\varepsilon,\mathcal{P}^\perp}^{\mathrm{b/f}}) \subseteq [-\alpha^2/4, +\infty)\,.$

For the study of the eigenvalues at the bottom of the spectrum (near $-\alpha^2$), the most relevant operator is $\widehat{H}^{\mathrm{b/f}}_{\varepsilon,\mathcal{P}}$.

We remark that unlike the smooth potential case, $\mathrm{ran}(\mathcal{P}|D(H^{\mathrm{b/f}}_{\varepsilon})) \not\subseteq D(H^{\mathrm{b/f}}_{\varepsilon})$, because $\psi^{BO} f_{\phi} \notin H^{\mathrm{b/f}}_{\varepsilon}$. For this reason, it is not possible to identify $\widehat{H}^{\mathrm{b/f}}_{\varepsilon,\mathcal{P}}$ with the compression $\mathcal{P} H^{\mathrm{b/f}}_{\varepsilon} \mathcal{P}$, which is not well defined.

- D. Krejčiřík, N. Raymond, J. Royer, and P. Siegl. Mathematika 2018.

Proposition. *For all* $\phi \in H^1_{\mathrm{b/f}}(\mathbb{R}^2)$ *there holds*

$$\left|\mathcal{B}^{\mathrm{b/f}}_{\varepsilon}(\phi) - \widehat{\mathcal{B}}^{\mathrm{b/f}}_{\varepsilon}(\phi)\right| \leq 2\varepsilon \|[\partial_x, \mathcal{P}]\| \left(\sqrt{\mathcal{B}^{\mathrm{b/f}}_{\varepsilon}(\phi) + \alpha^2} + \sqrt{\widehat{\mathcal{B}}^{\mathrm{b/f}}_{\varepsilon}(\phi) + \alpha^2}\right) \|\phi\|.$$

Note that

$$[\partial_x, \mathcal{P}]\phi = \left(\partial_x \psi^{BO}\right) f_\phi + \psi^{BO} \tilde{f}_\phi,$$

with

$$f_\phi(x) = \int_{\mathbb{R}} \psi^{BO}(x,y)\phi(x,y)\mathrm{d}y, \qquad \tilde{f}_\phi(x) := \int_{\mathbb{R}} \left(\partial_x \psi^{BO}(x,y)\right)\phi(x,y)\mathrm{d}y\,.$$

And on has the bound

$$\|\,[\partial_x, \mathcal{P}]\,\| \leq 2\left(\sup_{x\in\mathbb{R}} \int |\partial_x \psi^{BO}(x,y)|^2 \mathrm{d}y\right)^{1/2} \leq C|\alpha|.$$

The latter bound uses the explicit form of the function $\psi^{BO}(x,y)$.

Effective Hamiltonian for the heavy particles.

$$\mathcal{B}_\varepsilon^{\mathrm{b/f}}(\mathcal{P}\phi) = \int_{\mathbb{R}^2} \varepsilon^2 |\partial_x(\psi^{BO} f_\phi)|^2 \mathrm{d}\mathbf{x} + \int_{\mathbb{R}} b_x(\psi_x^{BO}) |f_\phi|^2 \, \mathrm{d}x.$$

Hence,

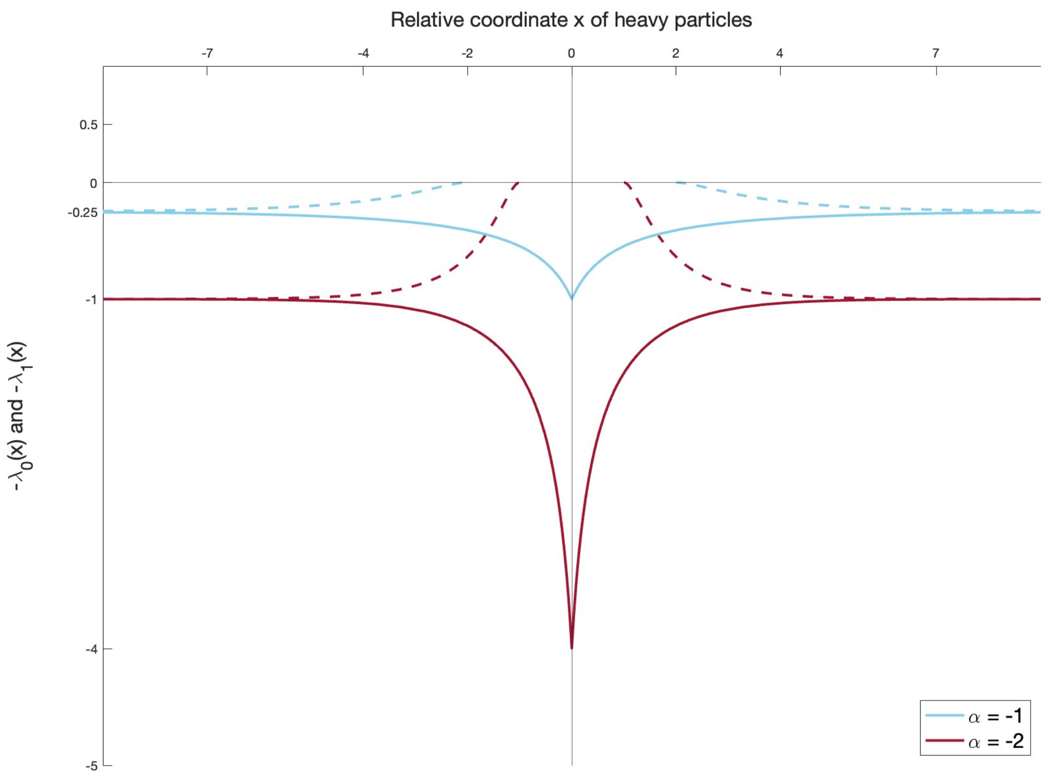
$$\mathcal{B}_\varepsilon^{\mathrm{b/f}}(\mathcal{P}\phi) = \int_{\mathbb{R}} \big(\varepsilon^2 |f_\phi'|^2 + \big(- \lambda_0(x) + \varepsilon^2 \, R\big) |f_\phi|^2 \, \mathrm{d}x$$

with

$$R(x) := \int_{\mathbb{R}} |\partial_x \psi^{BO}(x,y)|^2 \mathrm{d}y \, .$$

We have reduced the problem to the study of the effective Hamiltonian

$$\boxed{D(H_\varepsilon^{\mathrm{eff\,b/f}}) = H^2(\mathbb{R}) \cap L^2_{\mathrm{b/f}}(\mathbb{R}) \qquad H_\varepsilon^{\mathrm{eff\,b/f}} = -\varepsilon^2 \frac{d^2}{dx^2} - \lambda_0 + \varepsilon^2 \, R.}$$



$$\lambda_0 \in C^\infty(\mathbb{R}\backslash\{0\}), \qquad -\alpha^2 \le -\lambda_0(x) < -\alpha^2/4$$

We use IMS localization technique:

- R. S. Ismagilov. Dokl. Akad. Nauk SSSR 1961.
- J. D. Morgan. J. Operator Theory 1979.
- J. D. Morgan and B. Simon. Int. J. Quant. Chem. 1990.
- B. Simon. Ann. Inst. H. Poincaré 1983.

The potential term $-\lambda_0$ is only piecewise smooth, it is continuous, but not differentiable at $x = 0$.

$-\lambda_0$ is linear around $x = 0$, rather than quadratic, as in the smooth case:

$$-\lambda_0(x) = -\alpha^2 + |\alpha|^3|x| + O(x^2)\,, \qquad |x| \ll 1\,.$$

For this reason the eigenvalue expansion involves zeroes and extrema of the Airy function[2]

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[2] As already pointed out in Akbas, Turgut JMP 2018 and Duchêne, Raymond JPA 2014.

Theorem. *For any fixed integer $n \geq 0$ and $\varepsilon > 0$ sufficiently small, $H_\varepsilon^{\mathrm{eff\,b/f}}$ has at least $n+1$ simple isolated eigenvalues. The $(n+1)$-th eigenvalue is given by*

$$E_{\varepsilon,n}^{\mathrm{eff\,b/f}} = -\alpha^2 + s_n^{\mathrm{b/f}}\alpha^2\varepsilon^{2/3} + O(\varepsilon)$$

Essential spectrum

$R_\varepsilon^{\mathrm{b/f}}(-\lambda) = (H_\varepsilon^{\mathrm{b/f}} + \lambda)^{-1}$

$$R_\varepsilon^{\mathrm{b/f}}(-\lambda) = R_\varepsilon^0(-\lambda) - 2\alpha G_\varepsilon^{\mathrm{b/f}}(-\lambda)\big(1 + 2\alpha M_\varepsilon^{\mathrm{b/f}}(-\lambda)\big)^{-1} \breve{G}_\varepsilon^{\mathrm{b/f}}(-\lambda) \qquad \lambda > \alpha^2$$

$$M_\varepsilon^{\mathrm{b/f}}(-\lambda) = \frac{1}{2}\big(M_{d,\varepsilon}(-\lambda) \ \pm \ M_{od,\varepsilon}(-\lambda)\big) \qquad \lambda > 0$$

In Fourier transform, $M_{d,\varepsilon}(-\lambda)$ is the multiplication operator for

$$\hat{M}_{d,\varepsilon}(\nu; -\lambda) = \frac{1}{\sqrt{4\varepsilon^2\nu^2 + (4+\varepsilon^2)\lambda}}$$

and $M_{od,\varepsilon}(-\lambda)$ is the operator with integral kernel

$$\hat{M}_{od,\varepsilon}(\nu, \nu'; -\lambda) = \frac{2}{\pi}\frac{1}{(4+\varepsilon^2)(\nu^2 + \nu'^2) + 2(4-\varepsilon^2)\nu\nu' + 4\lambda}$$

$M_{d,\varepsilon}(-\lambda)$ is bounded and $M_{od,\varepsilon}(-\lambda)$ is Hilbert-Schmidt

Lemma. *Let* $\lambda > 0$. *Then,*

(i) $-\lambda \in \sigma_p(H_\varepsilon^{\mathrm{b/f}})$ *if and only if* $0 \in \sigma_p(1 + 2\alpha M_\varepsilon^{\mathrm{b/f}}(-\lambda))$;

(ii) $-\lambda \in \sigma_{ess}(H_\varepsilon^{\mathrm{b/f}})$ *if and only if* $0 \in \sigma_{ess}(1 + 2\alpha M_\varepsilon^{\mathrm{b/f}}(-\lambda))$.

- $[0, +\infty) \subseteq \sigma_{ess}(H_\varepsilon^\natural)$ by constructing singular Weyl sequences
- Additionally, setting

$$f_{\lambda,\varepsilon}(s) = 1 + \frac{\alpha}{\sqrt{4\varepsilon^2 s^2 + (4+\varepsilon^2)\lambda}}$$

$$\sigma_{ess}\big(1 + 2\alpha M_\varepsilon^{\mathrm{b/f}}(-\lambda)\big) = \text{range}(f_{\lambda,\varepsilon}) = \left[1 - \frac{|\alpha|}{\sqrt{(4+\varepsilon^2)\lambda}}\, , 1\right] .$$

Given $\lambda > 0$, $-\lambda \in \sigma_{ess}(H_\varepsilon^\natural)$ if and only if $0 \in \left[1 - \frac{|\alpha|}{\sqrt{(4+\varepsilon^2)\lambda}}\, , 1\right]$.

And so $\sigma_{ess}(H_\varepsilon^\natural) = \left[-\frac{\alpha^2}{4+\varepsilon^2}\, , 0\right) \cup [0, +\infty)$.

Perspectives

- BO approximation with point interactions in dimension 2.
    - An Hamiltonian in $L^2(\mathbb{R}^4)$ in the Jacobi coordinate $\mathbf{x} \in \mathbb{R}^2$ and $\mathbf{y} \in \mathbb{R}^2$.
    - Equivalent to the Laplacian $\Delta = \Delta_{\mathbf{x}} + \Delta_{\mathbf{y}}$ with a singular interaction supported on
$$\{\mathbf{y} = \varepsilon\mathbf{x}/2\} \cup \{\mathbf{y} = -\varepsilon\mathbf{x}/2\}$$
- BO approximation with point interactions in dimension 3.
    - It is plagued by the so-called fall to the center phenomenon Minlos - Faddeev 1962 and Thomas 1984
    - To keep the Hamiltonian bounded from below a renormalization of the interaction is needed
    - Basti - Ferretti - Teta 2026, assume the validity of the BO approximation and prove the Efimov effect.

Perspectives

- While not connected to quantum graphs, these problems are related to Leaky Quantum Structures (see Exner PSIM 2020):

  $$H_{\alpha,\Gamma} = -\Delta + \alpha\delta_\Gamma \qquad \alpha < 0, \text{ in } L^2(\mathbb{R}^n)$$

  here Γ is a geometric complex understood as a subset of $\mathbb{R}^d$.

- The approach developed in Krejčiřík, Raymond, Royer, and Siegl. Mathematika 2018 is very effective for:
  - the study of the BO approximation with smooth potentials
  - the study of problems in codimension 1, for the analysis of shrinking tubular neighborhoods of hypersurfaces

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