Constant sign and sign changing NLS ground states on metric graphs

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Joint work with

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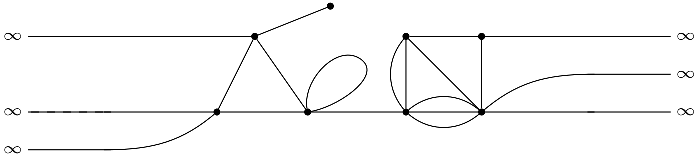
Metric graphs: informal definition

A metric graph $\mathcal{G} = (\mathbb{V}, \mathbb{E})$ is a connected network made up of edges $e \in \mathbb{E}$, glued at vertices $\mathrm{v} \in \mathbb{V}$, according to the topology of a graph:

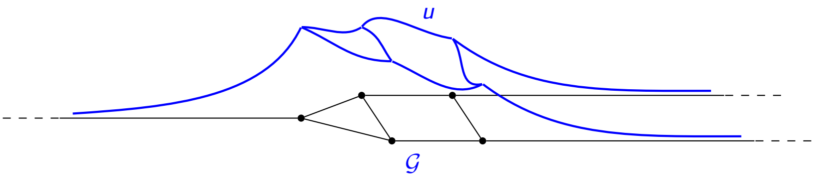


- We always assume $\mathcal{G}$ is connected and has an at most countable number of edges, $\deg(\mathrm{v}) < \infty$ for every $\mathrm{v} \in \mathbb{V}$, $\inf_{e \in \mathbb{E}} |e| > 0$, $\mathbb{V}_0 \subset \mathbb{V}$ is a (possibly empty) set of degree 1 vertices.
- Any bounded edge e is identified with an interval $[0, \ell_e]$, any unbounded one with a half–line $[0, +\infty)$
- With the shortest-path distance, we obtain a metric space $\mathcal{G}$
- $u \in L^p(\mathcal{G}) \iff \begin{cases} u \in L^p(e) & \text{for every edge } e \text{ of } \mathcal{G} \\ \sum_{e \in \mathbb{E}} \|u\|^p_{L^p(e)} < +\infty \end{cases}$

The Sobolev space $H^1(\mathcal{G})$ is defined as follows

$$u \in H^1(\mathcal{G}) \iff \begin{cases} u \in H^1(e) & \text{for every edge } e \text{ of } \mathcal{G} \\ u : \mathcal{G} \to \mathbb{R} & \text{is continuous on } \mathcal{G} \\ \sum_{e\in\mathbb{E}} \|u\|^2_{H^1(e)} < +\infty \end{cases}$$

Here is what a typical $H^1(\mathcal{G})$ function looks like:



The problem

On a metric graph $\mathcal{G}$ we consider the problem

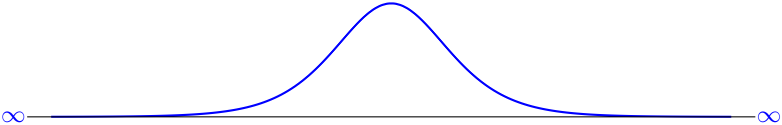
$$\begin{cases} -u'' + \lambda u = |u|^{p-2}u & \text{on every edge } e \in \mathbb{E} \quad (1) \\ u \text{ is continuous} & \text{on } \mathcal{G} \\ u(\mathrm{v}) = 0 & \text{at every vertex } \mathrm{v} \in \mathbb{V}_0 \quad (2) \\ \sum_{e \succ \mathrm{v}} u_e'(\mathrm{v}) = 0 & \text{at every vertex } \mathrm{v} \in \mathbb{V} \setminus \mathbb{V}_0 \quad (3) \end{cases}$$

where $p > 2$, $\lambda > 0$,
$u_e'(\mathrm{v})$ is the derivative away from the vertex v along the edge e
the sum $\sum_{e \succ \mathrm{v}}$ means that the sum is on all edges e incident at v,
$\mathbb{V}_0 \subset \mathbb{V}$ is made of vertex of degree 1.

(1) is the NLS equation,
(2) is the Dirichlet condition,
(3) is the Kirchhoff (or natural) condition.

Applications :
1) Propagation of signals in some optical fibers leads to a nonlinear Schrödinger equation (with $p = 4$). The cubic term therein comes from the Kerr effect, referring to a nonlinear change in the refraction index of an optical material.
2) Bose-Einstein condensates
3) ...

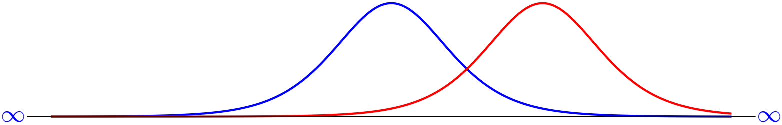
Exemple 1 - Travelling solitons



Note:

$$\phi_\lambda(x) = \left(\frac{p}{2}\right)^{\frac{1}{p-2}} \lambda^{\frac{1}{p-2}} \left(\cosh\left(\frac{p-2}{2}\sqrt{\lambda}\, x\right)\right)^{\frac{-2}{p-2}}$$

Exemple 1 - Travelling solitons



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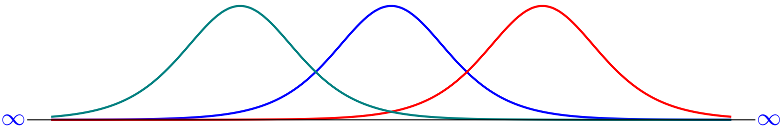
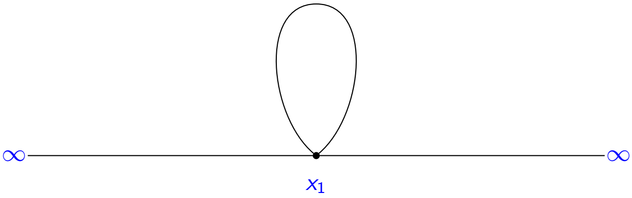


Example 2 - The real line with two points glued together



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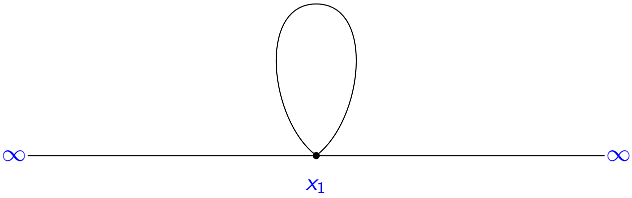
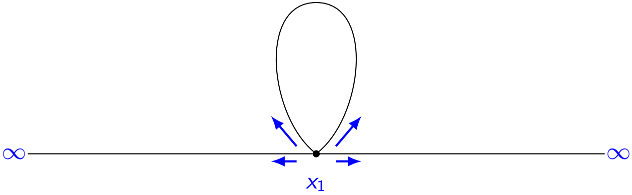
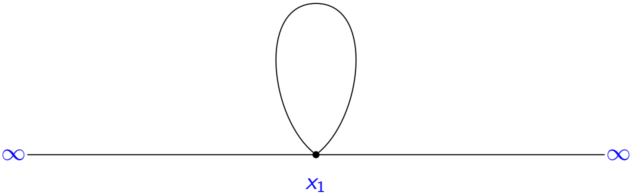


One solution is “easy” ,



$$\sum u_e'(x_1) = 0,$$



more solutions ?? "easy"

Phase Plane Analysis

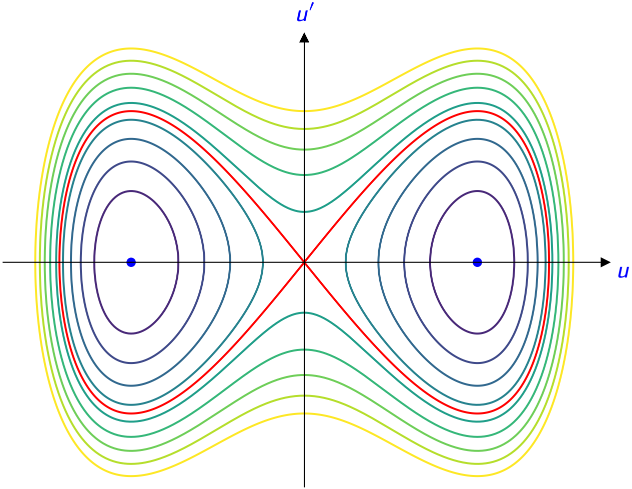
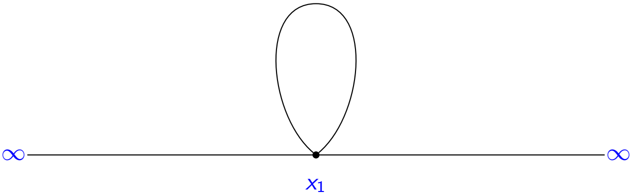
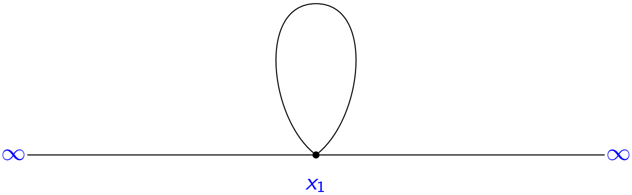


Figure: Phase portrait of the ODE when $\lambda > 0$



(Non-trivial) Solutions:

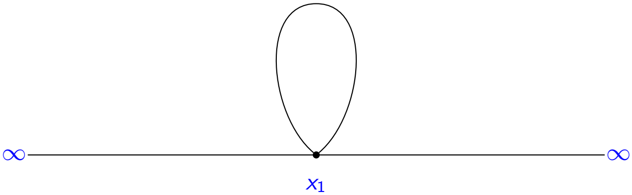
- The soliton “glued on all the graph”



(Non-trivial) Solutions:

• The soliton "glued on all the graph"

• The soliton on the two half-lines and a periodic solution on the loop, either positive (if the length of the loop is long enough) or sign-changing ... and some translate



(Non-trivial) Solutions:

• The soliton "glued on all the graph"

• The soliton on the two half-lines and a periodic solution on the loop, either positive (if the length of the loop is long enough) or sign-changing ... and some translate

• 0 on the two half-lines, a periodic sign-changing solution on the loop with 0 on the node.

"easy" 

Question 1: One vertex create so many different new situations ? How ?

How to prove existence ?

u is a *weak solution* if $\forall v \in H^1_D(\mathcal{G})$, $\displaystyle\int_{\mathcal{G}} u'v' + \lambda \int_{\mathcal{G}} uv - \int_{\mathcal{G}} |u|^{p-2}uv = 0$,

where $\displaystyle\int_{\mathcal{G}} = \sum_i \int_{e_i}$, the sum being on all the edges of $\mathcal{G}$ and

$$H^1_D(\mathcal{G}) = \{u \in \mathcal{C}(\mathcal{G}) \mid \int_{\mathcal{G}} |u'|^2 + \int_{\mathcal{G}} |u|^2 < \infty \quad \text{and} \quad u(\mathrm{v}) = 0,\ \forall \mathrm{v} \in \mathbb{V}_0\}.$$

This comes from the fact that, if $e_i = (a_i, b_i)$ and if u is regular enough

$$\int_{e_i} u'v' = -u'_{e_i}(b_i)v(b_i) - u'_{e_i}(a_i)v(a_i) - \int_{e_i} u''v$$

and hence, for all v,

$$\int_{\mathcal{G}} u'v' + \lambda \int_{\mathcal{G}} uv - \int_{\mathcal{G}} |u|^{p-2}uv = \sum_i \int_{e_i} u'v' + \lambda \sum_i \int_{e_i} uv - \sum_i \int_{e_i} |u|^{p-2}uv$$
$$= -\sum_{\mathrm{v}} \sum_{e \succ \mathrm{v}} u'_e(\mathrm{v})v(\mathrm{v}) - \int_{\mathcal{G}} u''v + \lambda \int_{\mathcal{G}} uv - \int_{\mathcal{G}} |u|^{p-2}uv$$

Hence we look for critical points of the action functional

$$J : H^1_D(\mathcal{G}) \to \mathbb{R} : J(u) := \frac{1}{2}\int_{\mathcal{G}} |u'|^2 + \frac{\lambda}{2}\int_{\mathcal{G}} |u|^2 - \frac{1}{p}\int_{\mathcal{G}} |u|^p$$

Behaviour of the action functional

We are interested in positive and sign changing solutions that "minimize" the action: positive and nodal ground states.

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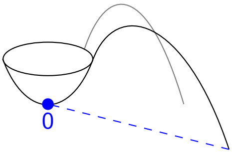

1) 0 is a local minimum but ... too simple
2) there is no global minimum

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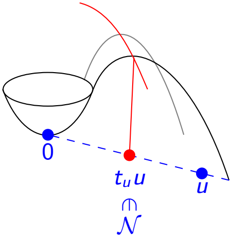


1) 0 is a local minimum but ... too simple
2) there is no global minimum

As J is not bounded from below on $H^1_D(\mathcal{G})$ we introduce the Nehari manifold and the nodal Nehari set associated with J:

$$\begin{aligned}\mathcal{N}(\mathcal{G}) &= \left\{u \in H^1_D(\mathcal{G}) \mid u \neq 0,\ J'(u)u = 0\right\} \\ &= \left\{u \in H^1_D(\mathcal{G}) \mid u \neq 0,\ \int_{\mathcal{G}} |u'|^2 + \lambda \int_{\mathcal{G}} |u|^2 = \int_{\mathcal{G}} |u|^p\right\}\end{aligned}$$

$$\mathcal{M}(\mathcal{G}) = \left\{u \in H^1_D(\mathcal{G}) \mid u^{\pm} \in \mathcal{N}(\mathcal{G})\right\}$$

Of course $\mathcal{M}(\mathcal{G}) \subset \mathcal{N}(\mathcal{G})$ and if $u \in \mathcal{N}(\mathcal{G})$, then

$$J(u) = \kappa \|u\|_{L^p(\mathcal{G})}^p = \kappa(\|u'\|_{L^2(\mathcal{G})}^2 + \lambda \|u\|_{L^2(\mathcal{G})}^2), \qquad \kappa = \frac{1}{2} - \frac{1}{p},$$

so that J is bounded from below on $\mathcal{N}(\mathcal{G})$ and $\mathcal{M}(\mathcal{G})$.

Definition

We say that

- $u \in H_D^1(\mathcal{G})$ is a (positive) (action) ground state if

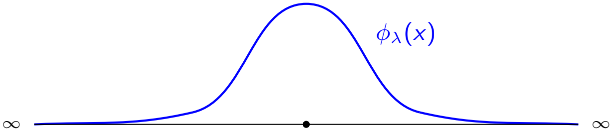
$$J(u) = \inf_{v \in \mathcal{N}(\mathcal{G})} J(v).$$

- $u \in H_D^1(\mathcal{G})$ is a (action) nodal ground state if

$$J(u) = \inf_{v \in \mathcal{M}(\mathcal{G})} J(v).$$

Fundamental example - real line

When $\mathcal{G}$ is the real line, it is well known that ground states exist. They are the family of solitons $\pm\phi_\lambda(x-c)$, $\quad c \in \mathbb{R}$



We denote by s_λ the soliton level:

$$s_\lambda := J(\phi_\lambda) = \inf_{v \in \mathcal{N}(\mathbb{R})} J(v).$$

On the contrary,

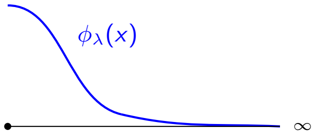
$$\inf_{v \in \mathcal{M}(\mathbb{R})} J(v) = 2s_\lambda$$

and it is not achieved: nodal ground states do not exist on $\mathbb{R}$.

Fundamental example - half-line

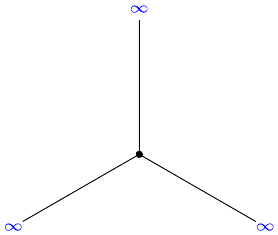
When $\mathcal{G}$ is the half-line, it is well known that

1. ground states exist if we put Neumann condition on the vertice
   The solutions are the half-solitons $\pm\phi_\lambda(x)$,

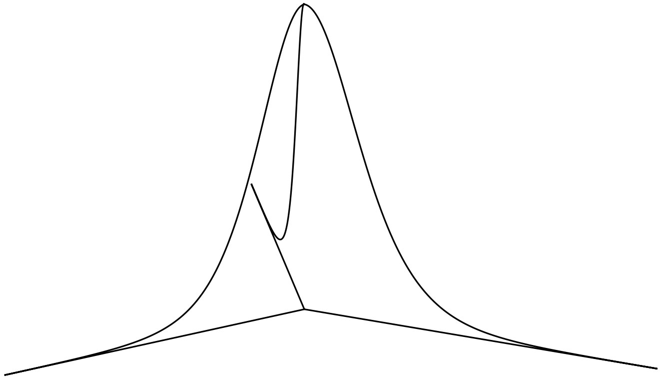


2. ground states do not exist if we put Dirichlet condition on the vertice.
3. nodal ground states do not exist on $\mathbb{R}^+$.

Fundamental example - 3-star-graph

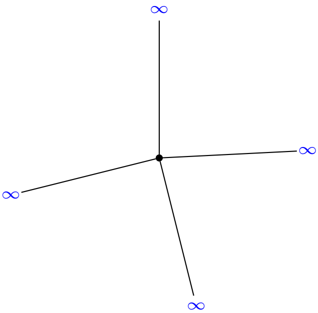


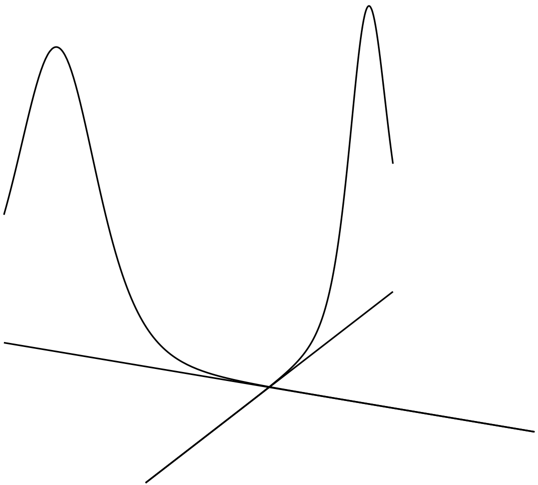
When $\mathcal{G}$ is the 3-star graph, we known that the solutions are given by



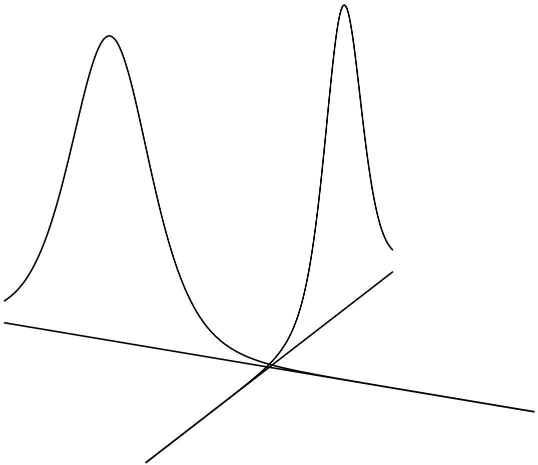
The positive solution on the 3-star graph

Fundamental example - 4-star-graph





A continuous family of solutions on the 4-star graph



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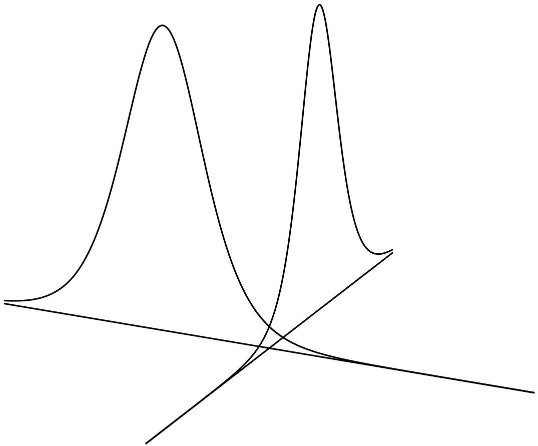

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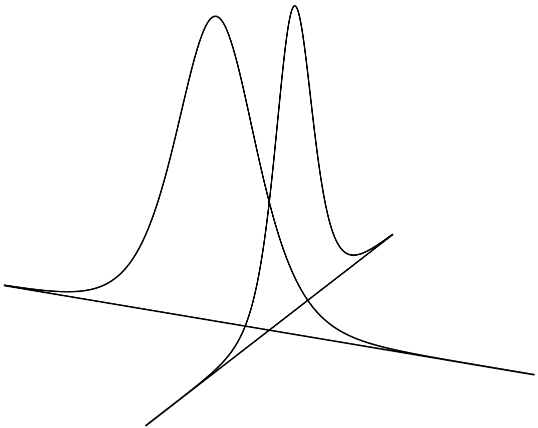

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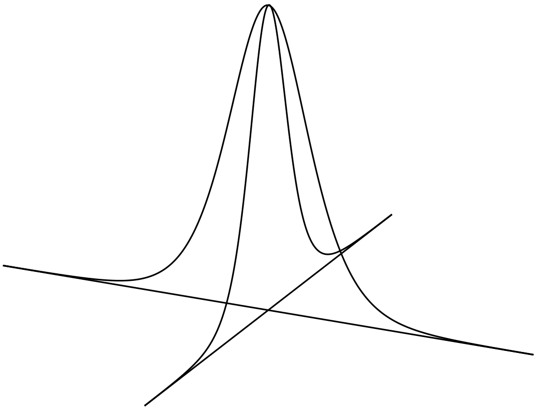

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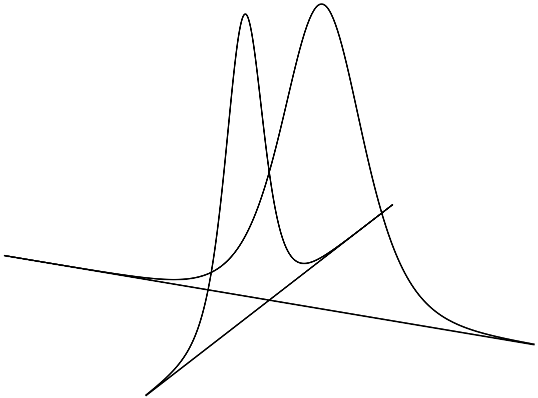

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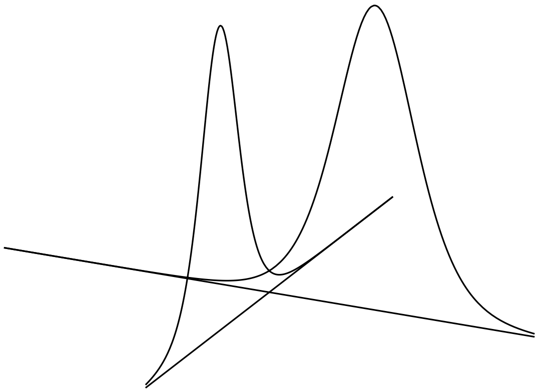

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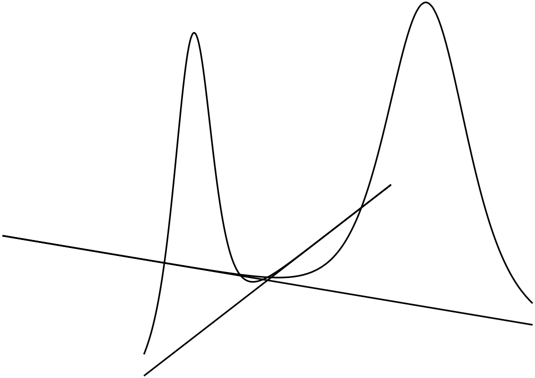

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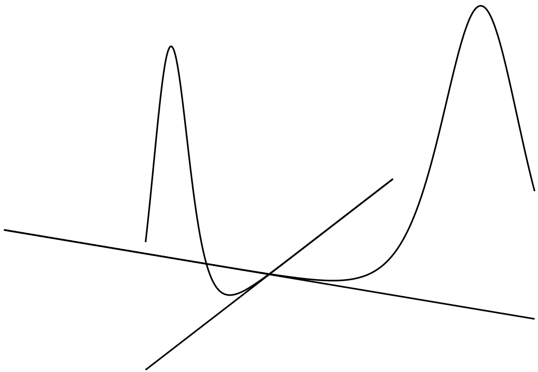

A continuous family of solutions on the 4-star graph

Fundamental example

On star graph (with $N \geq 3$):

1. No ground states.
2. No nodal ground states.
3. No nodal solutions.

What it's all about

On a compact graph $\mathcal{G}$ (finite number of edges of finite length), ground state and nodal ground state always exist.

A graph is characterized by topological and metrical properties.

Our aim is to inspect how these features can

- rule out ground states or nodal ground states

  or

- ensure their existence

working at a high level of generality

consider the impact of the boundary condition (Dirichlet somewhere or not)

The abstract way to existence

There is a somewhat standard way to study existence.
One defines the level at infinity

$$J^\infty(\mathcal{G}) := \inf\Big\{\liminf_n J(u_n) \mid (u_n)_n \subset \mathcal{N}(\mathcal{G}),\ u_n \rightharpoonup 0 \text{ in } H^1_D(\mathcal{G})\Big\}$$

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and proves

Theorem (Abstract existence theorem)

Let $\mathcal{G}$ be a noncompact graph.
If

$$\inf_{v\in\mathcal{N}(\mathcal{G})} J(v) < J^\infty(\mathcal{G}),$$

then $\mathcal{G}$ admits a ground state.

If

$$\inf_{v\in\mathcal{M}(\mathcal{G})} J(v) < J^\infty(\mathcal{G}) + \inf_{v\in\mathcal{N}(\mathcal{G})} J(v),$$

then $\mathcal{G}$ admits a nodal ground state.

Idea of the proof

We construct a min. sequence $(u_n)_n \subset \mathcal{N}(\mathcal{G})$ (resp. $(u_n)_n \subset \mathcal{M}(\mathcal{G})$) s.t.

$$u_n \rightharpoonup u \quad \text{weakly in } H^1(\mathcal{G}) \qquad \text{and} \qquad J'(u) = 0.$$

In the first case, we have to avoid that $u \equiv 0$. if $(u_n)_n \subset \mathcal{N}(\mathcal{G})$ is such that $u_n \rightharpoonup 0$ in $H^1_D(\mathcal{G})$,

$$\inf_{v \in \mathcal{N}(\mathcal{G})} J(v) = \liminf_{n\to\infty} J(u_n) \geq J^\infty(\mathcal{G}) \qquad \text{contradiction.}$$

In the second case, we have to show that $u^\pm \not\equiv 0$. For every n,

$$J(u_n) = J(u_n^+) + J(u_n^-) \geq J(u_n^+) + \inf_{v \in \mathcal{N}(\mathcal{G})} J(v).$$

If, for instance, $u^+ \equiv 0$, then $u_n^+ \rightharpoonup 0$ in $H^1_D(\mathcal{G})$, so that

$$\begin{aligned} \inf_{v \in \mathcal{M}(\mathcal{G})} J(v) &= \liminf_{n\to\infty} J(u_n) \geq \liminf_{n\to\infty} J(u_n^+) + \inf_{v \in \mathcal{N}(\mathcal{G})} J(v) \\ &\geq J^\infty(\mathcal{G}; Z) + \inf_{v \in \mathcal{N}(\mathcal{G})} J(v) \qquad \text{contradiction.} \end{aligned}$$

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Good. But if one is not able to compute $J^\infty(\mathcal{G})$, this is useless.

Noncompact graphs with a half-line

The behavior of noncompact graphs with an half-line is very different from that of graphs with infinitely many bounded edges.

In the first case the soliton level s_λ plays a central role.

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Theorem

Let $\mathcal{G}$ be a noncompact graph with at least one half-line. Then

$$\inf_{v \in \mathcal{N}(\mathcal{G})} J(v) \leq s_\lambda$$

$$\inf_{v \in \mathcal{M}(\mathcal{G})} J(v) \leq s_\lambda + \inf_{v \in \mathcal{N}(\mathcal{G})} J(v)$$

If moreover, $\mathcal{G}$ has a finite number of edges

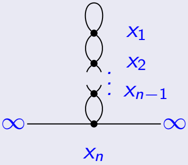
$$J^\infty(\mathcal{G}) = s_\lambda$$

Nonexistence by topological properties

Theorem

Let $\mathcal{G}$ have at least one half-line.

$$\sigma(\mathcal{G}) = 0 \quad \Longrightarrow \quad \begin{cases} \inf_{v \in \mathcal{N}(\mathcal{G})} J(v) = s_\lambda \\ \text{no ground state unless } \mathcal{G} \text{ is isometric to} \end{cases}$$



$$\sigma(\mathcal{G}) \leq 1 \quad \Longrightarrow \quad \begin{cases} \inf_{v \in \mathcal{M}(\mathcal{G})} J(v) = s_\lambda + \inf_{v \in \mathcal{N}(\mathcal{G})} J(v) \\ \mathcal{G} \text{ admits no nodal ground state} \end{cases}$$

The set of “vertices at infinity” of $\mathcal{G}$ is

$$\mathbb{V}_\infty = \{\mathrm{v} \text{ is the “vertex” of degree 1 of some half-line}\}.$$

Define the set

$$F(\mathcal{G}) = \Big\{ e \in \mathbb{E} \;\Big|\; \text{at least one connected component of } \mathcal{G} \setminus \{e\} \\ \text{has no vertices in } \mathbb{V}_0 \cup \mathbb{V}_\infty \Big\}.$$

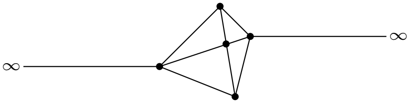
Definition

Let $\mathcal{G}$ be a graph with a finite number of edges. The splitting index $\sigma(\mathcal{G})$ of $\mathcal{G}$ is the cardinality of $F(\mathcal{G})$:

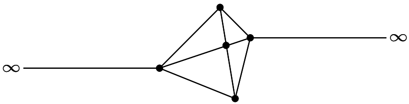
$$\sigma(\mathcal{G}) := \# F(\mathcal{G}).$$

The splitting index of $\mathcal{G}$ is a purely topological concept.

Graphs with $\mathbb{V}_0 = \emptyset$

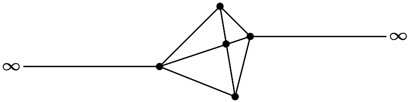


Graphs with $\mathbb{V}_0 = \emptyset$



$\sigma(\mathcal{G}) = 0$

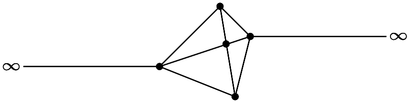
Graphs with $\mathbb{V}_0 = \emptyset$



$\sigma(\mathcal{G}) = 0$

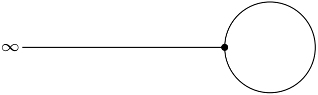
No GS, No NGS

Graphs with $\mathbb{V}_0 = \emptyset$

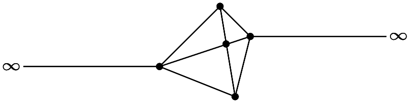


$\sigma(\mathcal{G}) = 0$

No GS, No NGS



Graphs with $\mathbb{V}_0 = \emptyset$

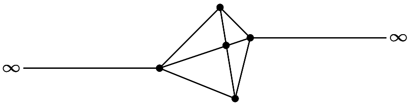


$\sigma(\mathcal{G}) = 0$

No GS, No NGS

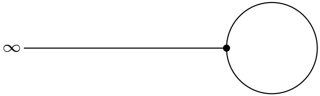
$\sigma(\mathcal{G}) = 1$

Graphs with $\mathbb{V}_0 = \emptyset$



$\sigma(\mathcal{G}) = 0$

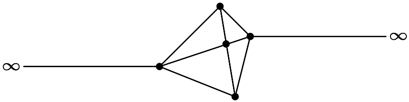
No GS, No NGS



$\sigma(\mathcal{G}) = 1$

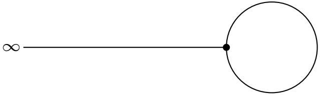
GS ?, No NGS

Graphs with $\mathbb{V}_0 = \emptyset$



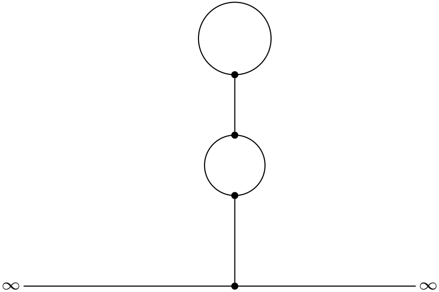
$\sigma(\mathcal{G}) = 0$

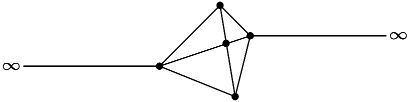
No GS, No NGS



$\sigma(\mathcal{G}) = 1$

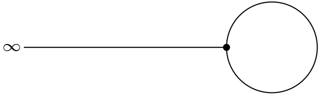
GS ?, No NGS





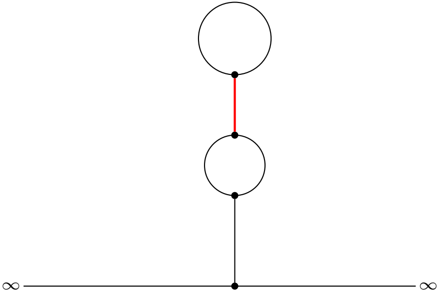
$\sigma(\mathcal{G}) = 0$

No GS, No NGS



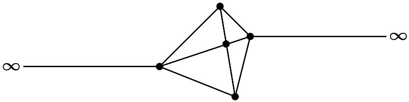
$\sigma(\mathcal{G}) = 1$

GS ?, No NGS



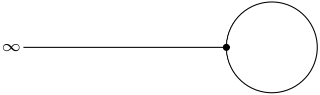
$\sigma(\mathcal{G}) = 2$

Graphs with $\mathbb{V}_0 = \emptyset$



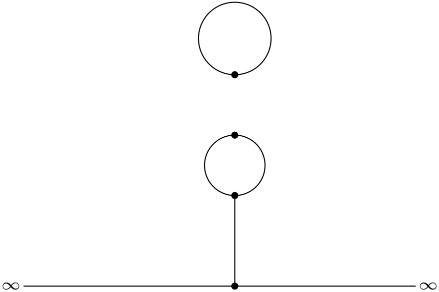
$\sigma(\mathcal{G}) = 0$

No GS, No NGS



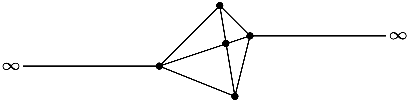
$\sigma(\mathcal{G}) = 1$

GS ?, No NGS



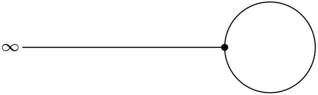
$\sigma(\mathcal{G}) = 2$

Graphs with $\mathbb{V}_0 = \emptyset$



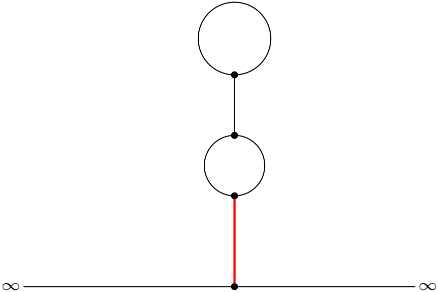
$\sigma(\mathcal{G}) = 0$

No GS, No NGS



$\sigma(\mathcal{G}) = 1$

GS ?, No NGS



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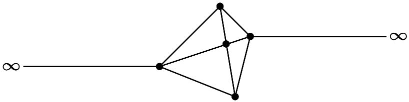
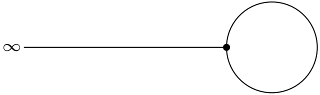
Graphs with $\mathbb{V}_0 = \emptyset$



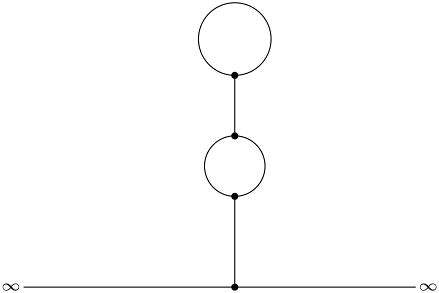
$\sigma(\mathcal{G}) = 0$

No GS, No NGS



$\sigma(\mathcal{G}) = 1$

GS ?, No NGS

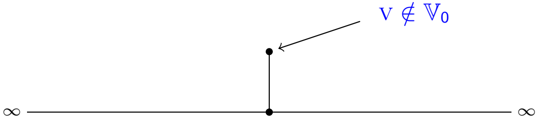


$\sigma(\mathcal{G}) = 2$

GS ?, NGS ?

Graphs with $\mathbb{V}_0 \neq \emptyset$

The presence of vertices $\mathrm{v} \in \mathbb{V}_0$ may lower $\sigma(\mathcal{G})$.



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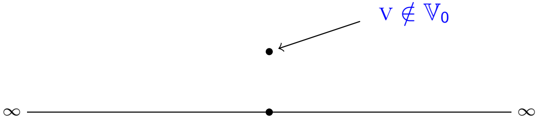

$$\sigma(\mathcal{G}) = 1$$

GS ?, No NGS

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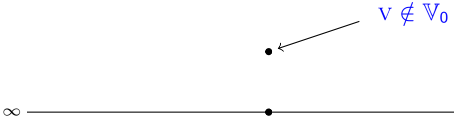
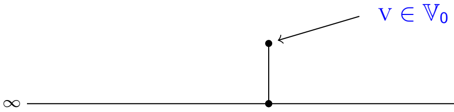


$$\sigma(\mathcal{G}) = 1$$

GS ?, No NGS



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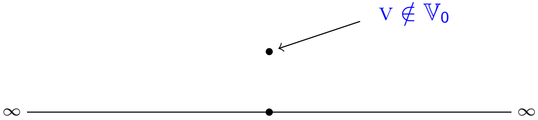
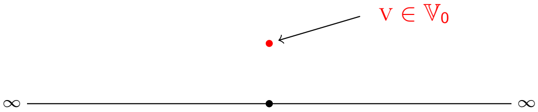


$\sigma(\mathcal{G}) = 1$

GS ?, No NGS



$\sigma(\mathcal{G}) = 0$

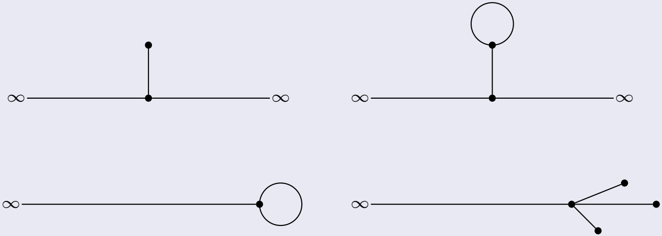
No GS, No NGS

Existence by topological properties

In a few cases the topology of the graph guarantees the existence of ground states.

Theorem

On each of the graphs



$\inf_{v \in \mathcal{N}(\mathcal{G})} J(v) < s_\lambda = J^\infty(\mathcal{G})$. *Hence they all admit ground states.*

Note that all these graphs satisfy $\sigma(\mathcal{G}) \geq 1$.
Therefore the nonexistence condition $\sigma(\mathcal{G}) = 0$ is sharp.

Existence by metrical properties

Here by "pendant" we mean a terminal edge whose vertex of degree 1 is not in $\mathbb{V}_0$.

Here by "pendant" we mean a terminal edge whose vertex of degree 1 is not in $\mathbb{V}_0$.

Theorem

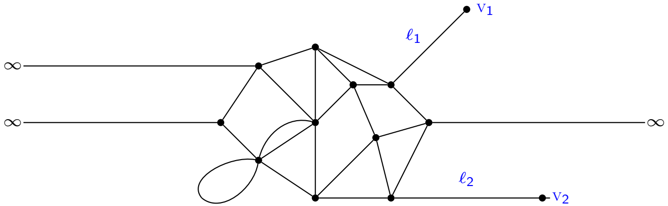
Let $\mathcal{G}$ be a noncompact graph with a finite number of edges.

If $\mathcal{G}$ has a sufficiently long pendant, then $\mathcal{G}$ admits a ground state.

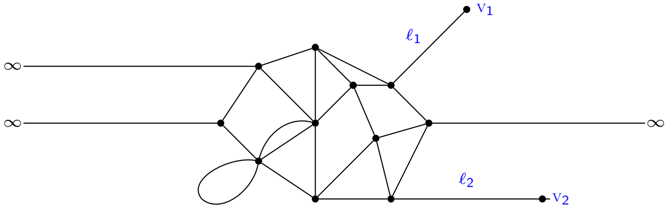
If $\mathcal{G}$ has two sufficiently long pendants, then $\mathcal{G}$ admits a nodal ground state.

Remark. The threshold on the lengths of the pendants depends on λ and p but not on $\mathcal{G}$.

A final example

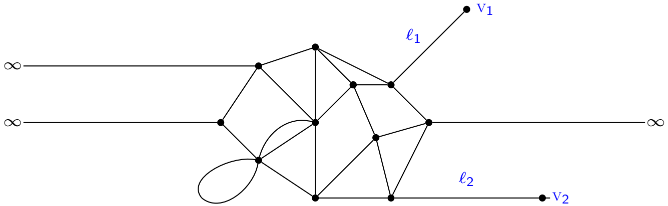


A final example



If $v_1, v_2 \in \mathbb{V}_0$, then $\sigma(\mathcal{G}) = 0$: no ground states, no nodal ground states

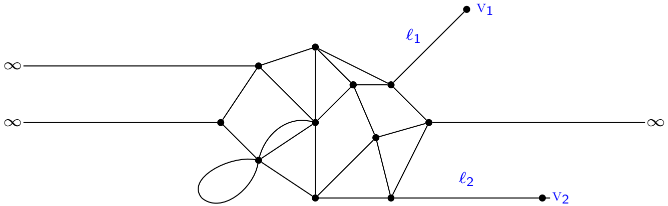
A final example



If $\mathrm{v}_1, \mathrm{v}_2 \in \mathbb{V}_0$, then $\sigma(\mathcal{G}) = 0$: no ground states, no nodal ground states

If $\mathrm{v}_1 \in \mathbb{V}_0$ and $\mathrm{v}_2 \notin \mathbb{V}_0$, then $\sigma(\mathcal{G}) = 1$: ground state if ℓ_2 large enough, no nodal ground states

A final example

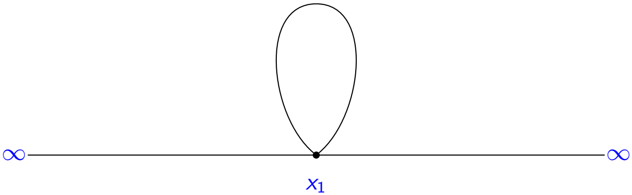


If $v_1, v_2 \in \mathbb{V}_0$, then $\sigma(\mathcal{G}) = 0$: no ground states, no nodal ground states

If $v_1 \in \mathbb{V}_0$ and $v_2 \notin \mathbb{V}_0$, then $\sigma(\mathcal{G}) = 1$: ground state if ℓ_2 large enough, no nodal ground states

If $v_1, v_2 \notin \mathbb{V}_0$, then $\sigma(\mathcal{G}) = 2$: ground state and nodal ground states if ℓ_1, ℓ_2 large enough

Questions

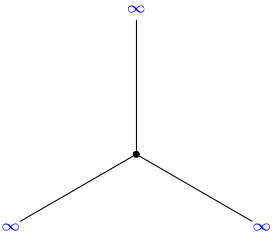


$$\sigma(\mathcal{G}) = 0$$

GS (the soliton "glued on all the graph"), no NGS

Question 2: Characterization of the known nodal solutions ?

Question 3: Characterization of the other positive solutions ?



$\sigma(\mathcal{G}) = 0$

no GS, no NGS

Question 4: Characterization of the solutions ?

Extension of the Nehari method

The Nehari method works well when $\lambda > -\lambda_1$ where $\lambda_1 = \inf \frac{\int_{\mathcal{G}} |u'|^2}{\int_{\mathcal{G}} |u|^2}$ in order to have the good behavior around 0.

Extension:

- **A. Pankov**, *Periodic nonlinear Schrödinger equation with application to photonic crystals*, Milan J. Math. 73 (2005), 259–287.
- **A. Szulkin and T. Weth**, *The Method of Nehari Manifold*, in Handbook of Nonconvex Analysis and Applications (D. Y. Gao and D. Motreanu, eds.), International Press Somerville, 597–632, 2010.

For graphs:

- **A. Pankov**, *Nonlinear Schrodinger equations of periodic metric graphs*, Discrete and continuous dynamical systems, 38, (2018), 697-714
- **S. Akduman, A. Pankov**, *Nonlinear Schrödinger equation with growing potential on infinite metric graphs*, Nonlinear Analysis 184 (2019) 258–272
- **D. Galant, T. Weth**, *Normalized solutions of Nehari-Pankov type to indefinite variational problems*, arxiv.2601.10941
- ...

Nehari "fixing the edge of the maximum"

In order to have other solutions than the AGS, we can fix the bounded edge where we wanted the solution has his maximum.

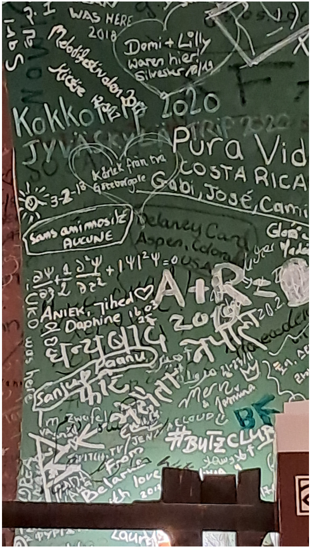
- **R. Adami, E. Serra, P. Tilli**, *Multiple positive bound states for the subcritical NLS equation on metric graphs*, Calc. Var. PDE 58(5), (2019) 16.
- **C. De Coster, S. Dovetta, D. Galant, E. Serra**, *On the notion of ground state for nonlinear Schrödinger equations on metric graphs*, Calc. Var. Partial Differential Equations, 62(2023) 159.
- **Q. Liu**, *Multiple positive bound states for NLS equations on noncompact metric graphs with an attractive potential*, arXiv:2602.18894v1
- **...**

Nehari “counting the number of roots”

- **C. Besse, R. Duboscq, S. Le Coz**, *Computation of excited states for the nonlinear Schrödinger equation: numerical and theoretical analysis*, ESAIM: M2AN 59 (2025), 899-923.
- **...**

- **L. Song**, *Properties of the least action level, bifurcation phenomena and the existence of normalized solutions for a family of semi-linear elliptic equations without the hypothesis of autonomy*, J. Differential Equations 315 (2022), 179–199.
- **L. Jeanjean, S.-S. Lu**, *On global minimizers for a mass constrained problem*, Calc. Var. Partial Differential Equations 61 (2022), no. 6, article no. 214
- **S. Dovetta, E. Serra, and P. Tilli**, *Action versus energy ground states in nonlinear Schrödinger equations*, Math. Ann. 385 (2023), no. 3-4, 1545–1576.
- **C. De Coster, S. Dovetta, D. Galant, E. Serra**, *An action approach to nodal and least energy normalized solutions for nonlinear Schrödinger equations*, Ann. Inst. H. Poincaré C Anal. Non Linéaire, published online first, 2025.
- **D. Galant, T. Weth**, *Normalized solutions of Nehari-Pankov type to indefinite variational problems*, arxiv.2601.10941
- **...**

- ...



Picture of a cafe of Stockholm, December 2025

Thanks Delio

Picture of Stockholm, December 2025

Thanks Delio