Nonlocal equations on co-evolving graphs

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Outline

Motivation & setup

Co-evolving graphs

Graph-continuity equation & long-time behaviour

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Co-evolving graphs

Graph-continuity equation & long-time behaviour

Why should we focus on evolutionary problems on graphs?

- Formation of biological networks (leaf venation networks, vascular and neural networks)

  G. Albi, M. Burger, J. Haskovec, P. Markowich and M. Schlottbom, *Active Particles, Volume 1*, 2017.
- Gas transport on networks/metric graphs

  M. Erbar, D. Forkert, J. Maas, D. Mugnolo, *Net. Heterog. Media*, 2022

  A. Fazeny, M. Burger, J.-F. Pietschmann, *EJAM*, 2025
- Data Science/Machine Learning: data representation as point clouds for clustering and classification

  M. Belkin, P. Niyogi, *Neural Comput.*, 2002

  R. R. Coifman, S. Lafon, *Appl. Comput. Harmon. Anal.*, 2006

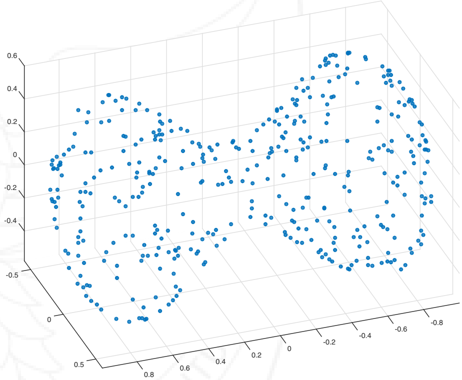
  K. Craig, N. Garcia-Trillos, N. Garcia, D. Slepcev, *Springer International Publishing*, 2022.

Pros & cons (good for us!)

- Easy/compact setting for: data representation, heterogeneity (e.g. interactions), etc.
- Non-standard setting: complicated geometry and analysis.

Dynamics on graphs: notation

- $X = \{x_1, x_2, \dots, x_n\}$ random sample i.i.d. according to $\mu \in \mathcal{M}^+(\mathbb{R}^d)$
  $\Rightarrow$ empirical measure $\mu^n = \frac{1}{n}\sum_{i=1}^n \delta_{x_i}$



Dynamics on graphs: notation

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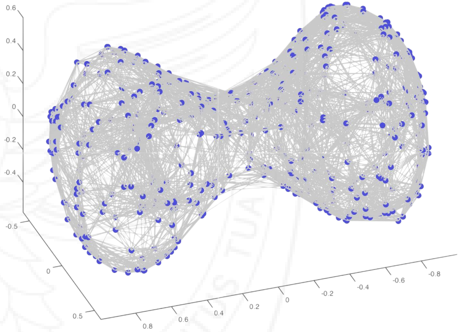
  $\Rightarrow$ empirical measure $\mu^n = \frac{1}{n}\sum_{i=1}^n \delta_{x_i}$
- a weight function $\eta : D \to \mathbb{R}$ with $D := (\mathbb{R}^d \times \mathbb{R}^d) \setminus \{x = y\}$
  $\Rightarrow (\mu^n, \eta)$ defines an discrete weighted graph



Example: dynamics driven by interaction energies on graphs

$$\mathcal{E}_X(\rho) = \frac{1}{2} \sum_{x \in X} \sum_{y \in X} K_{x,y} \rho_x \rho_y \tag{1}$$

On $\mathbb{R}^d$:

$$\dot{x}_i = -\sum_{j=1}^{n} \rho_j \nabla_x K(x_i, x_j) \tag{2}$$

On finite graphs

$$\frac{d\rho_x}{dt} = -\sum_{y \in X} j_{x,y} \eta(x,y) \tag{3}$$

$$j_{x,y} = \phi(\rho_x, \rho_y) v_{x,y} \tag{4}$$

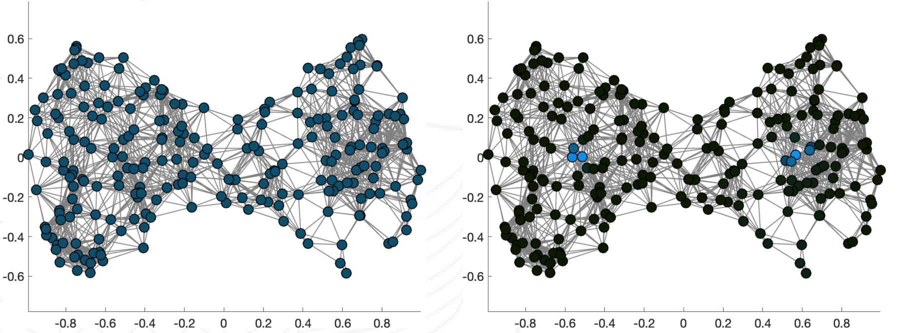
$$v_{x,y} = -\sum_{z \in X} \rho_z (K_{y,z} - K_{x,z}). \tag{5}$$

CHOICE IS NOT CANONICAL!

Goals

- Define “suitable” flows on graph $(\mu, \eta) \Rightarrow$ (co-)evolving if $\eta = \eta(t, \cdot)$
- Dynamics stable under graph limit $n \to \infty$ (discrete-to-continuum / many-vertices limit)
- Dynamics stable for local limit: $\mu = \mathrm{Leb}(\mathbb{R}^d)$, $\eta^\varepsilon(x,y) = \varepsilon^{-d-2} \eta\left(\frac{x-y}{\varepsilon}\right)$
  $\Rightarrow$ limit $\varepsilon \to 0$ should give $\partial_t \rho = \nabla \cdot (\rho \nabla K * \rho)$

Example: dynamics driven by interaction energies on graphs



General framework

- $\mathbb{R}^d$ set of possible vertices, $\mathbb{R}^d \times \mathbb{R}^d \setminus \{x = y\}$ set of possible edges
- $\eta : \mathbb{R}^d \times \mathbb{R}^d \setminus \{x = y\} \to \mathbb{R}$ weight function $\Rightarrow G := \{\mathbb{R}^d \times \mathbb{R}^d \setminus \{x = y\} | \eta(x, y) \neq 0\}$ set of edges
- $\mu \in \mathcal{M}^+(\mathbb{R}^d)$ set of vertices
- $\rho \in \mathcal{P}(\mathbb{R}^d)$ distribution of mass (or a measure $\mathcal{M}(\mathbb{R}^d)$)

Nonlocal continuity equation

Nonlocal continuity equation

Continuity equation

$$\partial_t \rho_t + \nabla \cdot j_t = 0 \qquad \text{where} \qquad j_t(x) := \rho_t(x) v_t(x)$$

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On Graphs

$$\partial_t \rho_t(x) + (\overline{\nabla} \cdot j_t)(x) = \partial_t \rho_t(x) + \int_{\mathbb{R}^d} j_t(x,y)\, \eta(x,y)\, d\mu(y) = 0$$

$$j_t(x,y) = \phi\big(\rho_t(x), \rho_t(y)\big)\, v_t(x,y)$$

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E. g. $\phi(r,s) = (r-s)/(\ln r - \ln s) \Rightarrow$ **not reasonable for repulsive interactions**

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Upwind interpolation: density along edges = density at the source

$$j_t(x,y) = \rho(x) v_t(x,y)_+ - \rho(y) v_t(x,y)_-$$

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Upwind interpolation: density along edges = density at the source

$$j_t(x,y) = \rho(x) v_t(x,y)_+ - \rho(y) v_t(x,y)_-$$

Nonlocal continuity equation ($\rho_t \ll \mu$)

$$\partial_t \rho_t(x) + \int_{\mathbb{R}^d} \left(\rho_t(x) v_t(x,y)_+ - \rho_t(y) v_t(x,y)_-\right) \eta(x,y)\, d\mu(y) = 0 \tag{NCE}$$

Nonlocal interaction equation on graphs: NL^{2}IE

$$\text{(NCE) with } v_t^{\mathcal{E}} := -\overline{\nabla} \frac{\delta \mathcal{E}}{\delta \rho} = -\overline{\nabla} K * \rho_t$$

A. E., F. S. Patacchini, A. Schlichting, D. Slepčev - ARMA (2021).

Gradient and divergence on a graph

Definition (Nonlocal gradient and divergence)

For any $\phi : \mathbb{R}^d \to \mathbb{R}$, we define its nonlocal gradient $\overline{\nabla}\phi : \mathbb{R}^{2d}_{\diagup} \to \mathbb{R}$ by

$$\overline{\nabla}\phi(x,y) = \phi(y) - \phi(x), \qquad \text{for all } (x,y) \in \mathbb{R}^{2d}_{\diagup}.$$

For any Radon measure $\boldsymbol{j} \in \mathcal{M}(\mathbb{R}^{2d}_{\diagup})$, its nonlocal divergence $\overline{\nabla} \cdot \boldsymbol{j} \in \mathcal{M}(\mathbb{R}^d)$ is defined as the adjoint of $\overline{\nabla}$, i.e., for any $\phi : \mathbb{R}^d \to \mathbb{R}$ in $C_0(\mathbb{R}^d)$, there holds

$$\begin{aligned}\int_{\mathbb{R}^d} \phi \,\mathrm{d}\overline{\nabla} \cdot \boldsymbol{j} &= -\frac{1}{2} \iint_{\mathbb{R}^{2d}_{\diagup}} \overline{\nabla}\phi(x,y) \mathrm{d}\boldsymbol{j}(x,y) \\ &= \frac{1}{2} \int_{\mathbb{R}^d} \phi(x) \int_{\mathbb{R}^d \setminus \{x\}} (\mathrm{d}\boldsymbol{j}(x,y) - \mathrm{d}\boldsymbol{j}(y,x)).\end{aligned}$$

In particular, for $\boldsymbol{j}$ antisymmetric, that is, $\boldsymbol{j} \in \mathcal{M}(\mathbb{R}^{2d}_{\diagup})$ and $\mathrm{d}\boldsymbol{j}(x,y) = -\mathrm{d}\boldsymbol{j}(y,x)$, denoted $\boldsymbol{j} \in \mathcal{M}^{\mathrm{as}}(\mathbb{R}^{2d}_{\diagup})$, we have

$$\int_{\mathbb{R}^d} \phi \,\mathrm{d}\overline{\nabla} \cdot \boldsymbol{j} = \iint_{\mathbb{R}^{2d}_{\diagup}} \phi(x) \mathrm{d}\boldsymbol{j}(x,y) \,.$$

General interpolations

$$\partial_t \rho + \overline{\nabla} \cdot F^{\Phi}[\mu, \eta; \rho_t, v_t] = 0 \tag{NCE}$$

Definition (Admissible flux interpolation)

A measurable function $\Phi\colon \mathbb{R}^3 \to \mathbb{R}$ is called an admissible flux interpolation provided that the following conditions hold:

(i) Φ satisfies

$$\Phi(0,0;v) = \Phi(a,b;0) = 0, \qquad \text{for all } a,b,v \in \mathbb{R}; \tag{1}$$

(ii) Φ is argument-wise Lipschitz in the sense that, for some $L_\Phi > 0$, any $a,b,c,d,v,w \in \mathbb{R}$, it holds

$$|\Phi(a,b;w) - \Phi(a,b;v)| \le L_\Phi(|a| + |b|)|w - v|; \tag{2a}$$

$$|\Phi(a,b;v) - \Phi(c,d;v)| \le L_\Phi(|a-c| + |b-d|)|v|; \tag{2b}$$

(iii) Φ is positively one-homogeneous in its first and second argument, that is, for all $\alpha > 0$ and $(a,b,w) \in \mathbb{R}^3$, it holds

$$\Phi(\alpha a, \alpha b; w) = \alpha \Phi(a,b;w).$$

General interpolations: examples

- Upwind interpolation. One important case is given by the upwind interpolation Φ_{upwind} defined as

$$\Phi_{\text{upwind}}(a, b; w) = aw_+ - bw_- \qquad \text{for } (a, b, w) \in \mathbb{R}^3. \tag{3}$$

- Mean multipliers. Another case is product interpolation Φ_{prod}, which is of the form

$$\Phi_{\text{prod}}(a, b; w) = \phi(a, b)w \qquad \text{for } (a, b, w) \in \mathbb{R}^3,$$

  with $\phi\, \mathbb{R}^2 \to \mathbb{R}$ any measurable function satisfying, for some $L_\Phi > 0$,

$$\begin{aligned} &|\phi(a, b)| \leq L_\Phi \max\{|a|, |b|\}, \\ &|\phi(a, b) - \phi(c, d)| \leq L_\Phi(|a - c| + |b - d|), \\ &\phi(\alpha a, \alpha b) = \alpha\phi(a, b), \\ &\phi(a, b) = \phi(b, a), \end{aligned}$$

  for all $\alpha \geq 0$ and $a, b, c, d \in \mathbb{R}$. Common choices for ϕ are as below:

  - *Arithmetic mean.* $\phi_{\text{AM}}(a, b) := \frac{a+b}{2}$;
  - *Maximal mean.* $\phi_{\max}(a, b) := \max\{a, b\}$.

Admissible fluxes and nonlocal conservation laws

Definition (Admissible flux)

Let Φ be an admissible flux interpolation, $\rho \in \mathcal{M}_{\mathrm{TV}}(\mathbb{R}^d)$ and $w \in \mathcal{V}^{\mathrm{as}}(G) := \{v\ G \to \mathbb{R} : v(x,y) = -v(y,x)\}$. Furthermore, take $\lambda \in \mathcal{M}^+(\mathbb{R}^d \times \mathbb{R}^d)$ such that $\rho \otimes \mu, \mu \otimes \rho \ll \lambda$ (e.g., $\lambda = |\rho| \otimes \mu + \mu \otimes |\rho|$). Then, the admissible flux $F^\Phi[\mu; \rho, w] \in \mathcal{M}(G)$ at (ρ, w) is defined by

$$\mathrm{d}F^\Phi[\mu, \eta; \rho, w] = \Phi\left(\frac{\mathrm{d}(\rho \otimes \mu)}{\mathrm{d}\lambda}, \frac{\mathrm{d}(\mu \otimes \rho)}{\mathrm{d}\lambda}; w\right) \eta \mathrm{d}\lambda. \tag{4}$$

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Definition (Measure-valued solution to the NCL)

Given Φ and a measurable $V : [0, T] \times \mathcal{M}_{\mathrm{TV}}(\mathbb{R}^d) \to \mathcal{V}^{\mathrm{as}}(G)$, a curve $\rho : [0, T] \to \mathcal{M}_{\mathrm{TV}}(\mathbb{R}^d)$ is a measure-valued solution to the NCL, denoted as

$$\partial_t \rho + \overline{\nabla} \cdot F^\Phi[\mu, \eta; \rho, V_t(\rho)] = 0, \tag{NCL}$$

provided that, for any $A \in \mathcal{B}(\mathbb{R}^d)$, it holds that

(i) $\rho \in \mathcal{AC}_t$;

(ii) $t \mapsto \overline{\nabla} \cdot F^\Phi[\mu; \rho_t, V_t(\rho_t)][A] \in L^1([0, T])$;

(iii) ρ satisfies

$$\rho_t[A] + \int_0^t \overline{\nabla} \cdot F^\Phi[\mu; \rho_s, V_s(\rho_s)][A] \mathrm{d}s = \rho_0[A] \qquad \text{for a.e. } t \in [0, T]. \tag{5}$$

Well-posedness

Theorem (A. E., F. S. Patacchini, A. Schlichting, EJAM ’23)

Let $V\colon [0,T]\times\mathcal{M}_{\mathrm{TV}}^{M}(\mathbb{R}^d)\to\mathcal{V}^{\mathrm{as}}(G)$ *and* $\exists\, C_V, L_V>0$ *so that, for all* $t\in[0,T]$ *and all* $\rho,\sigma\in\mathcal{M}_{\mathrm{TV}}^{M}(\mathbb{R}^d)$,

$$\sup_{x\in\mathbb{R}^d}\int_{\mathbb{R}^d\setminus\{x\}}|V_t[\rho](x,y)|\eta(x,y)d\mu(y)\le C_V,$$

$$\sup_{x\in\mathbb{R}^d}\int_{\mathbb{R}^d\setminus\{x\}}|V_t[\rho](x,y)-V_t[\sigma](x,y)|\eta(x,y)d\mu(y)\le L_V\|\rho-\sigma\|_{TV}.$$

Then, there exists a unique measure solution ρ *to* (NCL) *such that* $\rho_0=\rho^0$.

1. *Proof via Banach Fixed-Point Theorem*
2. $\Phi\equiv\Phi_{\mathrm{upwind}}$: GF approach for NL^{2}IE by generalised Wasserstein distance $\Rightarrow$ Finsler GF

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Co-evolving graphs

$$\begin{aligned}\partial_t \rho_t &= -\overline{\nabla} \cdot F^\Phi[\mu, \eta_t; \rho_t, V_t[\rho_t]],\\ \partial_t \eta_t &= \omega[\rho_t] - \eta_t,\end{aligned} \tag{Co-NCL}$$

$$\mathrm{d}F^\Phi[\mu, \eta; \rho, w] = \Phi\left(\frac{\mathrm{d}(\rho\otimes\mu)}{\mathrm{d}\lambda}, \frac{\mathrm{d}(\mu\otimes\rho)}{\mathrm{d}\lambda}; w\right)\eta\,\mathrm{d}\lambda.$$

Co-evolving graphs

$$
\begin{aligned}
\partial_t \rho_t &= -\overline{\nabla} \cdot F^{\Phi}[\mu, \eta_t; \rho_t, V_t[\rho_t]], \\
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\end{aligned}
\tag{Co-NCL}
$$

$$
\mathrm{d}F^{\Phi}[\mu, \eta; \rho, w] = \Phi\left(\frac{\mathrm{d}(\rho \otimes \mu)}{\mathrm{d}\lambda}, \frac{\mathrm{d}(\mu \otimes \rho)}{\mathrm{d}\lambda}; w \right) \eta \, \mathrm{d}\lambda.
$$

Definition (Solution to (Co-NCL))

Given Φ, $V : [0, T] \times \mathcal{M}_{\mathrm{TV}}(\mathbb{R}^d) \times \mathbb{R}^{2d}_{\diagup} \to \mathcal{V}^{as}(\mathbb{R}^{2d}_{\diagup})$, and $\omega : [0, T] \times \mathcal{M}_{\mathrm{TV}}(\mathbb{R}^d) \times \mathbb{R}^{2d}_{\diagup} \to \mathbb{R}$, a pair $(\rho, \eta) : [0, T] \to \mathcal{M}_{\mathrm{TV}}(\mathbb{R}^d) \times C_b(\mathbb{R}^{2d}_{\diagup})$ is a solution to the initial value problem (Co-NCL) if, for any $\varphi \in C_0(\mathbb{R}^d)$,

1. $\rho \in AC([0, T], \mathcal{M}_{\mathrm{TV}}(\mathbb{R}^d))$, $\eta \in AC([0, T], C_b(\mathbb{R}^{2d}_{\diagup}))$;
2. the maps $t \mapsto \langle \varphi, \overline{\nabla} \cdot F^{\Phi}[\mu, \eta_t; \rho_t, V_t[\rho_t]] \rangle$ and $t \mapsto \omega[\rho_t] - \eta_t \in L^1([0, T])$;
3. for a.e. $t \in [0, T]$, every $(x, y) \in \mathbb{R}^{2d}_{\diagup}$, for any $\varphi \in C_0(\mathbb{R}^d)$, it holds

$$
\int_{\mathbb{R}^d} \varphi \mathrm{d}\rho_t = \int_{\mathbb{R}^d} \varphi \mathrm{d}\rho_0 + \frac{1}{2} \int_0^t \iint_{\mathbb{R}^{2d}_{\diagup}} \overline{\nabla} \varphi \mathrm{d}F^{\Phi}[\mu, \eta_s, \rho_s; V_s[\rho_s]] \mathrm{d}s \tag{6}
$$

$$
\eta_t(x, y) = \eta_0(x, y) + \int_0^t \left(\omega[\rho_s](s, x, y) - \eta_s(x, y) \right) \mathrm{d}s. \tag{7}
$$

A.E., L. Mikolás, *On evolution PDEs on co-evolving graphs*, DCDS '24.

Different time-scales

Graph slower: $\tau = \varepsilon t$

$$\begin{cases} \partial_t \rho_t = -\overline{\nabla} \cdot F^{\Phi}[\mu, \eta_t; \rho_t, V_t[\rho_t]] \\ \partial_t \eta_t = \varepsilon(\omega[\rho_t] - \eta_t) \\ \rho_0 \in \mathcal{M}^M_{TV}(\mathbb{R}^d),\ \eta_0 \in C_b(\mathbb{R}^{2d}_{\diagup}), \end{cases} \qquad (\mathrm{Co-NCL_S})$$

Graph faster: $\tau = t/\varepsilon$

$$\begin{cases} \partial_t \rho_t = -\overline{\nabla} \cdot F^{\Phi}[\mu, \eta_t; \rho_t, V_t[\rho_t]] \\ \varepsilon \partial_t \eta_t(x,y) = -\eta_t(x,y) + \omega[\rho](t,x,y), \end{cases} \qquad (\mathrm{Co-NCL_F})$$

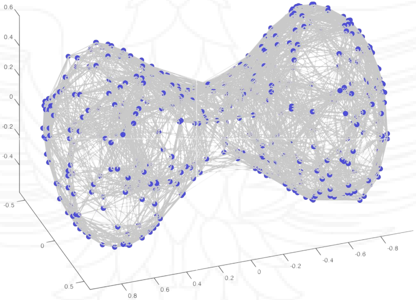
$\Rightarrow$

$$\partial_t \rho_t = -\overline{\nabla} \cdot F^{\Phi}[\mu, \omega[\rho_t]; \rho_t, V_t[\rho_t]].$$

Graph-limit or many-vertices limit

Finte graph

$$\mu^n = \sum_{i=1}^{n} m_i^n \delta_{x_i}\,, \quad x_i \in \mathbb{R}^d,\ m_i^n \in (0,\infty),\ \text{for } i = 1,\dots,n,\ \text{for } n \in \mathbb{N}. \tag{8}$$



$$\begin{cases} \partial_t \rho_t^n = -\overline{\nabla} \cdot F^{\Phi}[\mu^n, \eta_t^n; \rho_t^n, V_t[\rho_t^n]], \\ \partial_t \eta_t^n(x,y) = \omega_t[\rho^n](x,y) - \eta_t^n(x,y), \\ \rho_0 \in \mathcal{M}_{TV}^{M}(\mathbb{R}^d),\ \eta_0 \in C_b(\mathbb{R}^{2d}). \end{cases} \tag{Co$-$NCL$_n$}$$

Graph-limit or many-vertices limit: stability w.r.t. base measure μ

Theorem

Fix $\Phi \equiv \Phi_{upwind}$ *and consider a sequence* $\{\mu_n\}_{n\in\mathbb{N}} \in \mathcal{M}^+_{TV}(\mathbb{R}^d)$ *such that* $\mu^n \overset{*}{\rightharpoonup} \mu \in \mathcal{M}^+_{TV}(\mathbb{R}^d)$. *Let* $V : [0,T] \times \mathcal{M}^M_{TV}(\mathbb{R}^d) \times \mathbb{R}^{2d} \to \mathcal{V}^{as}(\mathbb{R}^{2d})$ *and* $\omega : [0,T] \times \mathcal{M}^M_{TV}(\mathbb{R}^d) \times \mathbb{R}^{2d} \to \mathbb{R}^d$ *satisfy suitable assumptions, uniformly in* n. *Assume* $((x,y) \mapsto V[\cdot](\cdot,x,y)) \in C_0(\mathbb{R}^{2d})$ *and* $((x,y) \mapsto \omega_t[\cdot](\cdot,x,y)) \in C_0(\mathbb{R}^{2d})$. *Let us consider a sequence of solutions* $\{(\rho^n,\eta^n)\}_{n\in\mathbb{N}}$ *to* $(\mathrm{Co-NCL_n})$ *associated to* $\{\mu_n\}$ *and let* (ρ,η) *be the solution to* (Co-NCL) *depending on* μ. *If* $\|\rho^n_0 - \rho^0\|_{TV} \to 0$ *and* $\|\eta^n_0 - \eta_0\|_\infty \to 0$ *as* $n \to \infty$, *then*

$$\lim_{n\to\infty} d_\infty((\rho^n,\eta^n),(\rho,\eta)) = 0\,.$$

$$d_\infty((\rho^1,\eta^1),(\rho^2,\eta^2)) := \|\rho^1-\rho^2\|_{\infty,\mathcal{M}_{TV}(\mathbb{R}^d)} + \|\eta^1-\eta^2\|_{\infty,C_b(\mathbb{R}^{2d})},$$

where

$$\|\rho^1-\rho^2\|_{\infty,\mathcal{M}_{TV}(\mathbb{R}^d)} = \sup_{t\in[0,T]} \|\rho^1_t-\rho^2_t\|_{\mathrm{TV}} \quad \text{and} \quad \|\eta^1-\eta^2\|_{\infty,C_b(\mathbb{R}^{2d})} = \sup_{t\in[0,T]} \|\eta^1_t-\eta^2_t\|_\infty .$$

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Euler equation for (Co-NCL)

Assume $\mu \in \mathcal{P}_2(\mathbb{R}^d)$ is the “set of vertices” $\Rightarrow \rho \ll \mu$, that is $\rho_t := r_t \mathrm{d}\mu$

$$\begin{aligned} \partial_t r_t(x) &= -\int_{\mathbb{R}^d\setminus\{x\}} \Phi(r_t(x), r_t(y); V_t[r](x,y))\eta_t(x,y)\mathrm{d}\mu(y) \\ \partial_t \eta_t &= \omega_t[r] - \eta_t, \end{aligned} \tag{Euler Co-NCL}$$

Definition (Solution to (Euler Co-NCL))

Let $p = 2, \infty$, Φ jointly antisym., $V : [0,T] \times L^p_\mu(\mathbb{R}^d) \to \mathcal{V}^{as}(\mathbb{R}^{2d}_{\diagup})$, and $\omega : [0,T] \times L^p_\mu(\mathbb{R}^d) \times \mathbb{R}^{2d}_{\diagup} \to \mathbb{R}$. A pair $(r, \eta) : [0,T] \to L^p_\mu(\mathbb{R}^d) \times C_b(\mathbb{R}^{2d}_{\diagup})$ is a sol to (Euler Co-NCL) w/ $(r_0, \eta_0) \in L^p_\mu(\mathbb{R}^d) \times C_b(\mathbb{R}^d)$ if:

1. $t \mapsto \int_{\mathbb{R}^d\setminus\{x\}} \Phi(r_t(x), r_t(y); V_t[r](x,y))\eta_t(x,y)\mathrm{d}\mu(y)$ and $t \mapsto \omega_t[\rho_t] - \eta_t$ belong to $L^1([0,T])$;
2. for μ-a.e. $x \in \mathbb{R}^d$, $t \in [0,T]$, $(x,y) \in \mathbb{R}^{2d}_{\diagup}$, it holds

$$r_t(x) = r_0(x) - \int_0^t \int_{\mathbb{R}^d\setminus\{x\}} \Phi(r_s(x), r_s(y); V_s[r](x,y))\eta_s(x,y)\mathrm{d}\mu(y)\mathrm{d}s \tag{9a}$$

$$\eta_t(x,y) = \eta_0(x,y) + \int_0^t \left(\omega_s[r](x,y) - \eta_s(x,y)\right) \mathrm{d}s. \tag{9b}$$

3. $r \in AC([0,T], L^p_\mu(\mathbb{R}^d))$, $\eta \in AC([0,T], C_b(\mathbb{R}^{2d}_{\diagup}))$.

Graph continuity equation

$\mathcal{G}_t = (\mu, \eta_t[r])$ co-evolving graph, for (r, η) sol to (Euler Co-NCL) $\Rightarrow \sigma := \delta_r \otimes \mu$ is a weak sol to

$$\begin{cases} \partial_t \sigma + \partial_\xi (\sigma \chi[\sigma, r, \eta]) = 0, \\ \sigma_0 = \bar{\sigma} \in \mathcal{P}(\mathbb{R} \times \mathbb{R}^d), \end{cases} \tag{graph-CE}$$

$$\chi[\sigma, r, \eta](t, \xi, x) := - \int_{\mathbb{R} \times \mathbb{R}^d \setminus \{x\}} \Phi(\xi, \xi'; V_t[r](x, x'))\eta_t(x, x')\mathrm{d}\sigma(\xi', x')\,, \tag{10}$$

Disintegrated version of (graph-CE)

$$\begin{cases} \partial_t \sigma_{t,x} + \partial_\xi (\sigma_{t,x} \chi[\sigma, r, \eta](t, \cdot, x)) = 0, \\ \sigma_{0,x} = \bar{\sigma}_x, \end{cases} \tag{11}$$

for μ-almost every $x \in \mathbb{R}^d$ and $\bar{\sigma} = \int_{\mathbb{R}^d} \bar{\sigma}_x \mathrm{d}\mu(x)$.

Theorem

Let $V : [0, T] \times L^2_\mu(\mathbb{R}^d) \to \mathcal{V}^{as}(\mathbb{R}^{2d}_{\diagup})$ *satisfy some assumptions. Let* (r, η) *be the solution of* (Euler Co-NCL). *Then, there exists a unique solution* $\sigma \in C([0, T], \mathcal{P}^\mu_2(\mathbb{R} \times \mathbb{R}^d))$ *to* (graph-CE) *with initial condition* $\nu \in \mathcal{P}^\mu_2(\mathbb{R} \times \mathbb{R}^d)$ *given by* $\sigma = \int_{\mathbb{R}^d} \sigma_{t,x} \mathrm{d}\mu(x)$ *where* $\sigma_{t,x} = f_x[\sigma, r, \eta](t, \cdot)_\# \nu_x$ *for* μ*-a.e.* $x \in \mathbb{R}^d$.

Inspired by Paul and E. Trélat - arXiv (2024) - similar case for heterogeneous interactions

Failure to prove contraction in $L^2_\mu d_2$: too strong condition on the flux

Setting

$$\mathcal{P}^\mu_2(\mathbb{R}\times\mathbb{R}^d) := \{\nu \in \mathcal{P}_2(\mathbb{R}\times\mathbb{R}^d) \mid \pi_{2\#}\nu = \mu\}\,,$$

$$L^2_\mu d_2(\nu^1,\nu^2) := \left(\int_{\mathbb{R}^d} d_2^2(\nu^1_x,\nu^2_x)\mathrm{d}\mu(x)\right)^{\frac{1}{2}}, \quad \nu^1,\nu^2 \in \mathcal{P}^\mu_2(\mathbb{R}\times\mathbb{R}^d)\,,$$

Proposition

Fix $\Phi \equiv \Phi_{upwind}$. *Let* $\sigma^i \in C([0,T],\mathcal{P}^\mu_2(\mathbb{R}\times\mathbb{R}^d))$ *be* μ*-monokinetic solutions of* (graph-CE), *with initial data* $\sigma^i_0 = \delta_{r^i_0(\cdot)} \otimes \mu$, *where* (r^i,η) *are solutions to* (Euler Co-NCL) *starting from* $(r^i_0,\bar\eta) \in L^2_\mu(\mathbb{R}^d) \times C_b(\mathbb{R}^{2d}_{\diagup})$, *for* $i = 1,2$, *and* $\bar\eta > 0$. *Let* $V : [0,T] \to \mathcal{V}^{as}(\mathbb{R}^{2d}_{\diagup})$ *be an antisymmetric velocity field which is continuous in time and satisfies suitable assumptions, and let* $\omega : [0,T]\times\mathbb{R}^{2d}_{\diagup} \to \mathbb{R}$ *satisfy other assumptions and* $\omega \geq \omega_* > 0$. *Assume,*

$$\lambda := \inf_{t\in[0,T]}\inf_{x\in\mathbb{R}^d}\int_{\mathbb{R}^d} V_t(x,x')\eta_t(x,x')\mathrm{d}\mu(x') > 0. \tag{12}$$

Then, for any $t \in [0,T]$,

$$L^2_\mu d_2(\sigma^1_t,\sigma^2_t) \leq e^{-2\lambda t} L^2_\mu d_2(\sigma^1_0,\sigma^2_0)\,.$$

Long-time behaviour for pointwise and monotonic velocities

Definition

A pointwise $V : [0, T] \times L^\infty_\mu(\mathbb{R}^d) \to \mathcal{V}^{as}(\mathbb{R}^{2d}_{\not\triangle})$ is *monotonic* if, for $r \in L^\infty_\mu(\mathbb{R}^d)$, $t \geq 0$, and μ-a.e. $x, x' \in \mathbb{R}^d$:

- $r(x) > r(y)$, then $V_t^+(r(x), r(x')) > V_t^+(r(y), r(x'))$,
- $r(x) < r(y)$, then $V_t^-(r(x), r(x')) > V_t^-(r(y), r(x'))$,
- $r(x) = r(x')$, then $V_t(r(x), r(x')) = 0$.

Main example

$$V_t(r_t(x), r_t(x')) = -\overline{\nabla}\alpha(r_t)(x, x') = \alpha[r_t(x)] - \alpha[r_t(x')] \, , \tag{13}$$

where $\alpha : \mathbb{R} \to \mathbb{R}$ is monotonically increasing and bounded such as $\alpha(x) = \frac{1}{1+e^{-x}}$, for $x \in \mathbb{R}^d$.

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- $r(x) > r(y)$, then $V^+_t(r(x), r(x')) > V^+_t(r(y), r(x'))$,
- $r(x) < r(y)$, then $V^-_t(r(x), r(x')) > V^-_t(r(y), r(x'))$,
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Assumption (restrict to K compact)

$V : [0, \infty) \times L^\infty_\mu(K) \to \mathcal{V}^{as}(K^2_{\diagup})$ is continuous in time and is of the form $V(r_t(x), r_t(x')) = \alpha(r_t(x)) - \alpha(r_t(x'))$, where $\alpha \in C^1_b(\mathbb{R})$ is increasing on compact sets, i.e.

$$\alpha'(y) = \frac{d}{dy}\alpha(y) > c > 0\,, \qquad \text{for } y \in \Omega \subset \mathbb{R} \text{ compact, and } c > 0.$$

Convergence to the uniform mass distribution

Theorem

Fix $\Phi \equiv \Phi_{upwind}$. *Let* $(r,\eta) \in AC(\mathbb{R}^+, L^\infty_\mu(K)) \times AC(\mathbb{R}^+, C_b(K^2_{\diagup}))$ *be a solution of* (Euler Co-NCL) *with a pointwise monotonic velocity field* $V : \mathbb{R}^+ \times L^\infty_\mu(K) \to \mathcal{V}^{as}(K^2_{\diagup})$ *satisfying suitable assumptions. Let* $\omega : \mathbb{R}^+ \times L^\infty_\mu(K) \times K^2_{\diagup} \to \mathbb{R}$ *suitable. Assume* $r_0 \geq 0$ *and* $\int_K r_0(x)\mathrm{d}\mu(x) = M$, *and* $\eta^0 \in C_b(K^2_{\diagup})$ *is positive and symmetric. Then, for* μ*-a.e.* $x \in \mathbb{R}^d$, *it holds*

$$\lim_{t\to\infty} r_t(x) = \frac{M}{\mu(K)}.$$

Proof

$$\begin{aligned}
\frac{r_{0,*}\eta_*\alpha'_{*,0}Me^{\eta_*\alpha'_{*,\mathbf{0}}Mt}}{\eta_*\alpha'_{*,0}(M+(e^{\eta_*\alpha'_{*,\mathbf{0}}Mt}-1)\mu(K)r_{0,*})} &\leq r_{t,*} \\
&\leq r_t(x) \\
&\leq \|r_t\|_{L^\infty_\mu(K)} \\
&\leq \frac{\|r_0\|_{L^\infty_\mu(K)}\alpha'_{*,0}\eta_*Me^{\alpha'_{*,\mathbf{0}}\eta_*Mt}}{\alpha'_{*,0}\eta_*\left[\|r_0\|_{L^\infty_\mu(K)}\mu(K)(e^{\alpha'_{*,\mathbf{0}}\eta_*Mt}-1)+M\right]}.
\end{aligned}$$

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- Evolution (nonlocal) PDEs on graphs (static and co-evolving)

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Thank you for your attention!