

On the stability of the JKO operator under different notions of convexity

mSPACE kick-off meeting, Bocconi University, Milan

Sara Farinelli,
Università di Genova & MaLGa

Joint work in progress with S. Di Marino and E. Naldi

Introduction to the proximal map

$E : \mathbb{R}^d \rightarrow \mathbb{R}$ smooth. Consider its **Gradient Flow**:

$$\begin{cases} \dot{x}(t) = -\nabla E(x(t)), \\ x(0) = x_0. \end{cases}$$

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Key point of De Giorgi's **minimizing movements** theory.

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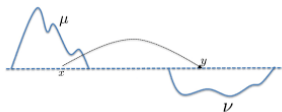
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2-Wasserstein distance

Optimal transport problem:

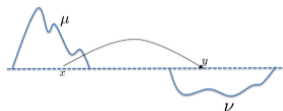
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For $\mu, \nu \in \mathcal{P}(\mathbb{R}^d)$,

$$\text{Adm}(\mu, \nu) = \left\{ \gamma \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d) : \pi_{\#}^1 \gamma = \mu, \pi_{\#}^2 \gamma = \nu \right\},$$

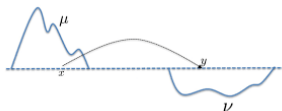
for $\pi^1 : (x, y) \mapsto x$, $\pi^2 : (x, y) \mapsto y$.

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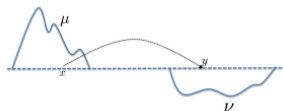
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- can be used to study **gradient flows** in the Wasserstein space, starting from a discretized problem.

Motivation I

On the big picture:

Open problem

Is the JKO operator of a functional $\mathcal{F} : \mathcal{P}_2(\mathbb{R}^d) \rightarrow (-\infty, +\infty]$ non expansive?

$$W_2(P_{\tau\mathcal{F}}(\mu), P_{\tau\mathcal{F}}(\nu)) \leq W_2(\mu, \nu) \quad \forall \mu, \nu \in \mathcal{P}_2(\mathbb{R}^d).$$

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Comment:

- $P_{\tau\mathcal{F}}$ can be used iteratively to define the discretized W_2 -gradient flow;
- for continuous time gradient flow non expansivity holds under convexity conditions of \mathcal{F} ! [Ambrosio, Gigli and Savaré '05].

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contractivity of the proximal operator \rightsquigarrow convergence of the proximal point algorithm.

- Variational analysis point of view:

$$\min_{\mu} \frac{1}{2} W_2^2(\mu, \nu) + \mathcal{F}(\mu),$$

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- **Evolutionary PDEs:** crowd motion with constrained density, continuity used to have uniqueness of the Wasserstein projection onto measures with bounded densities.

Convexity in the Wasserstein space

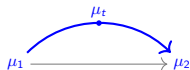
$\mathcal{F} : \mathcal{P}_2(\mathbb{R}^d) \rightarrow (-\infty, +\infty]$ is:

- geodesically convex if

$\forall \mu_1, \mu_2, \exists t \mapsto \mu_t$ geodesics between them s.t.

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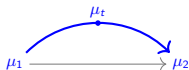
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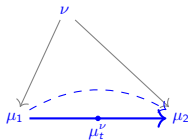
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s.t. $t \mapsto \mathcal{F}(\mu_t^\nu)$ is convex, [Ambrosio, Gigli and Savaré '05],

$\gamma_{\mu_i}^\nu \in \text{Opt}(\nu, \mu_i), \gamma \in \mathcal{P}((\mathbb{R}^d)^3)$ extending $\gamma_{\mu_i}^\nu,$

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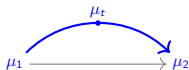
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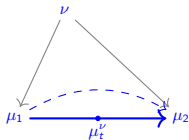
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- totally convex if

$\forall \mu_1, \mu_2, \forall$ curves induced by coupling

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Known cases

Total
convexity

Conv. along
gen. geod.

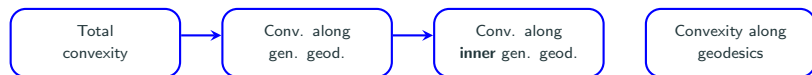
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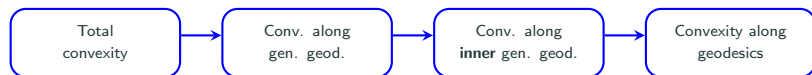
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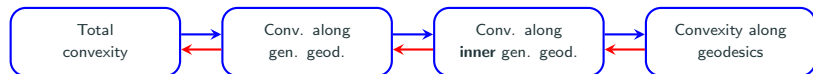
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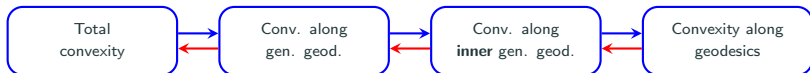
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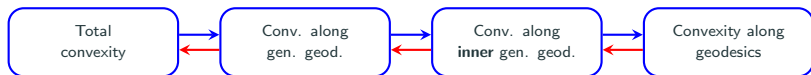


- Negative case: convexity along inner generalized geodesics is not enough for continuity:

$$K = \left\{ \mu \in \mathcal{P}_2(\mathbb{R}^d), \mu = \frac{1}{2}\delta_{x_1} + \frac{1}{2}\delta_{x_2} \right\}, \quad I_K(\nu) = \begin{cases} 0 & \text{if } \nu \in K, \\ +\infty & \text{else.} \end{cases}$$

Then P_{I_K} is the projection on K :
it is not even unique!

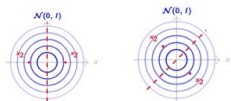
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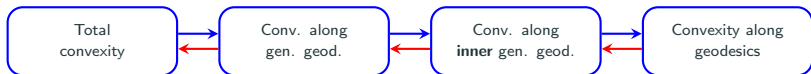
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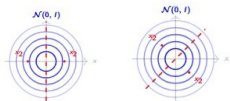
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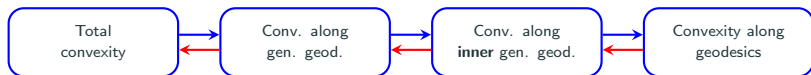
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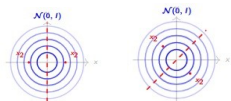
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Follows by lifting the totally convex functional to a convex functional on the Hilbert space of L^2 random variables.

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- [Carlen and Craig '13] : \mathcal{F} convex along generalized geodesics, then $P_{\tau\mathcal{F}}$ is non expansive w.r.t. a variation on W_2 ,

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Existing results:

- [Carlen and Craig '13] : \mathcal{F} convex along generalized geodesics, then $P_{\tau\mathcal{F}}$ is non expansive w.r.t. a variation on W_2 ,
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- $\frac{1}{2}$ -Hölderianity estimate was proved for a similar variational problem related to *moment measures*. [Delalande and F. '25]

Intermediate results II: weak non expansiveness

Existing results:

- [Adve and Meszáros '20]: Let $\Omega \subseteq \mathbb{R}^d$ closed convex set.

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Let $\mu, \nu \in \mathcal{P}_2(\Omega)$, and set $\tilde{\mu} := P_{\tau I_K}(\mu)$ and $\tilde{\nu} := P_{\tau I_K}(\nu)$. Then

$$W_2^2(\tilde{\mu}, \tilde{\nu}) \leq \int_{\Omega \times \Omega} |x - y|^2 d\gamma(x, y),$$

where

- $\eta \in \text{Opt}(\tilde{\mu}, \tilde{\nu})$,
- $T_\mu, T_\nu : \Omega \rightarrow \Omega$ the optimal maps from $\tilde{\mu}, \tilde{\nu}$ to μ, ν , respectively,
- $\gamma := (T_\mu, T_\nu)_\# \eta \in \text{Adm}(\mu, \nu)$.

$$\begin{array}{ccc} \mu & \xrightarrow{\gamma} & \nu \\ T_\mu \uparrow \text{opt} & \tilde{\eta} & \text{opt} \uparrow T_\nu \\ \tilde{\mu} & \xrightarrow[\text{opt}]{} & \tilde{\nu} \end{array}$$

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- if there exists a unique optimal transport plan between μ and $\nu \rightsquigarrow$ non expansiveness;

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- if there exists a unique optimal transport plan between μ and $\nu \rightsquigarrow$ non expansiveness;
- in general does not imply continuity.

Convex order of measures and totally convex functionals

Convex order on measures

Given $\mu, \nu \in \mathcal{P}(\mathbb{R}^d)$, μ and ν are in convex order $\mu \leq_C \nu$ if $\int_{\mathbb{R}^d} f \, d\mu \leq \int_{\mathbb{R}^d} f \, d\nu$ for any f continuous, convex and in $L^1(\nu)$.

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Proposition [Di Marino, F. and Naldi]

Let $\mathcal{F} : \mathcal{P}_2(\mathbb{R}^d) \rightarrow (-\infty, +\infty]$ proper, lower semicontinuous and totally convex. Then for any $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$ satisfying $\mu \leq_C \nu$, we have

$$\mathcal{F}(\mu) \leq \mathcal{F}(\nu).$$

Corollary

Let $\mathcal{F} : \mathcal{P}_2(\mathbb{R}^d) \rightarrow (-\infty, +\infty]$ proper, lower semicontinuous and totally convex. Then

$$\mathcal{F}(\delta_{\text{bar}(\mu)}) \leq \mathcal{F}(\mu) \quad \text{forevery } \mu \in \mathcal{P}(\mathbb{R}^d).$$

In particular, if $\mu \in \arg \min \mathcal{F}$ it also holds $\delta_{\text{bar}(\mu)} \in \arg \min \mathcal{F}$.

Thank you for your attention!