Recent advances in hyperuniformity

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Multiscale Stochastics, Patterns,
and Analysis of Combinatorial Environments

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Randomness vs Order vs Balance

Chaotic life (Poisson)

some days awesome, some days disaster, long-term instability.

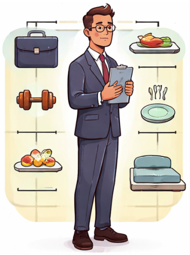
Rigid life (lattice)

Same routine every day, no spontaneous decisions

Hyperuniform life

spontaneous on everyday basis, stable on long-term

Chaotic Life



Ultra-Rigid Life

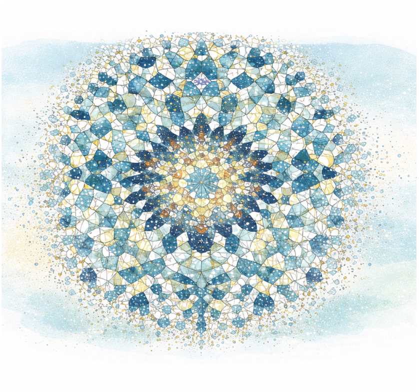
Balanced (Hyperuniform) Life

Nobel prize in Chemistry: Quasicrystals

Dan Shechtman (Technion, Israel): 2011 Nobel Prize for discovering quasicrystals.

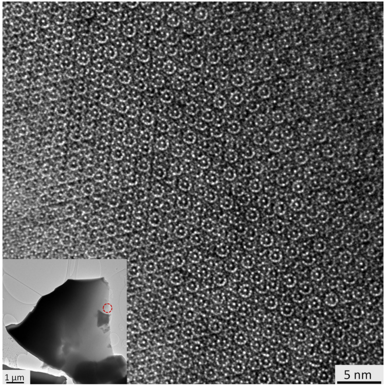
Quasicrystals:

- a unique form of matter found in 1982 exhibiting
  - long-range order,
  - no periodic repetition,
  - suppressed large-scale density fluctuations.
- They provide a beautiful example of structure *between* disorder and crystalline rigidity.

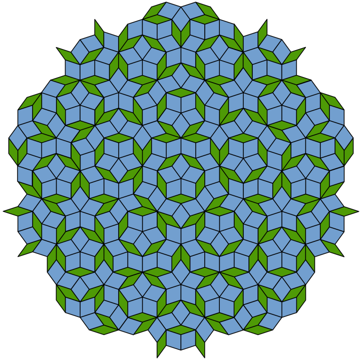


Quasicrystals: order beyond periodicity

Meteorit Khatyrka fragment



Penrose tilings



Outline

How to measure the *balance*?

Balance $\mapsto$ Hyperuniformity

local disorder, global order

- Hyperuniform systems suppress **long-range density fluctuations** more strongly than generic disordered media.
- They interpolate between randomness and rigidity, appearing in point processes, random media, tilings, and physical materials.
- The subject connects probability, geometry, statistical mechanics, and wave transport.

Takeaway

Hyperuniformity is a large-scale phenomenon: what matters is not local regularity, but the **asymptotic decay of fluctuations = variance.**

Probabilistic perspective

Hyperuniform point processes

Let $\mathcal{P}$ be a stationary point process in $\mathbb{R}^d$ with intensity ρ. For a bounded Borel set $W \subset \mathbb{R}^d$, denote by $N(W)$ the number of points in W.

Definition

The process $\mathcal{P}$ is **hyperuniform** if for a family of expanding windows W_R one has

$$\mathrm{Var}(N(W_R)) = o(|W_R|) \qquad \text{as } R \to \infty.$$

Problem: Definition strongly dependent on the shape of W_R.

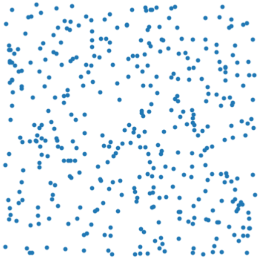
Common practice: $W_R = B(0, R)$

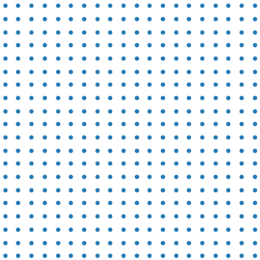
Examples

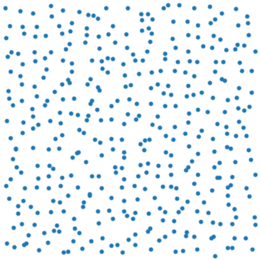
Non-hyperuniform
- Poisson
- Short-range Gibbs
- Perturbed lattices
- RSA (car parking)

Deterministic
- Lattices
- Crystals

Hyperuniform
- Perturbed lattices
- Zeros of Gaussian analytic functions
- Coulomb gas







Hyperuniformity $\Longrightarrow$ global repulsivity forces

Fourier characterization of hyperuniformity

Let $\mathcal{P}$ be a stationary point process of intensity ρ in $\mathbb{R}^d$. Its **structure factor** is defined by

$$S(k) = 1 + \rho\,\widehat{h}(k),$$

where $h = g_2 - 1$ is the total correlation function.

Definition

The process $\mathcal{P}$ is called **hyperuniform** if

$$S(k) \to 0 \qquad \text{as } |k| \to 0.$$

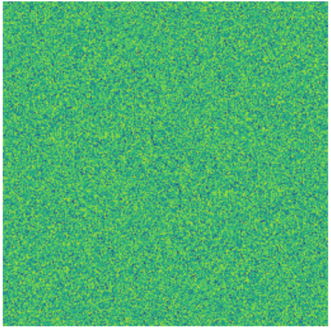
Problem: Often h does not exist nor admit Fourier transform.

Common practice: Work with distributions

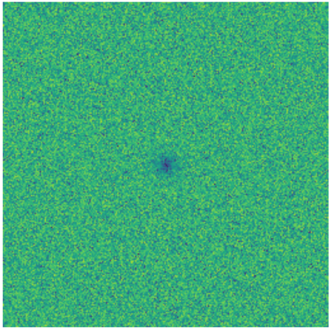
- **Open:** Construct a statistical test for hyperuniformity

Structure factor

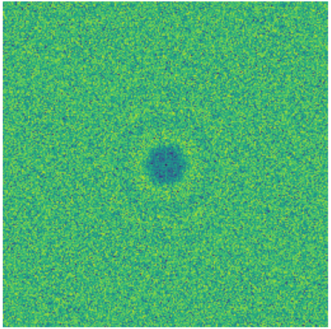
Poisson



perturbed lattice

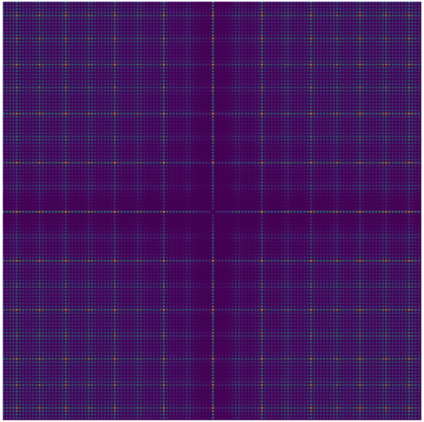


blue noise (stealthy hyperuniform)

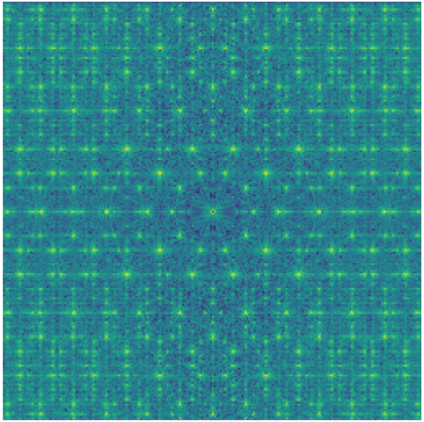


Structure factor

Crystal (lattice)



quasicrystal (Penrose tiling)



Spectral theory perspective

Spectral view - networks derived from hyperuniform patterns

Many physical systems can be modeled as **metric networks**:

- photonic crystals
- acoustic and mechanical metamaterials
- quantum networks

Key question

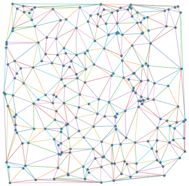
- How does hyperuniform geometry influence the spectrum of operators defined on the structure?
- Can one define hyperuniform spectral distributions?

Spectral view - networks derived from hyperuniform patterns

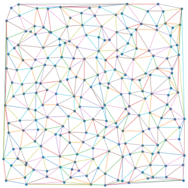
Hyperuniform point pattern	→	**Tessellation** (Delaunay, Gabriel)	→	**Metric network**	→	**Laplacian spectrum**

A **metric network** is a graph where each edge is assigned a length. Functions live on edges, differential operators act along edges. The Laplacian spectrum determines how waves propagate through the network.

Non-hyperuniform point pattern



Hyperuniform point pattern



Recent advances

Spectral problem

Let $\Delta = \frac{d^2}{dx^2}$ be the Laplacian with suitable vertex condfitions. Find eigenvalues λ and eigenfunctions ψ such that

$$-\Delta\psi = \lambda\psi.$$

Maher–Marzuola–Newhall (2026):

- Networks derived from hyperuniform patterns can develop gaps in the spectrum of the Laplacian.
- Increasing geometric disorder makes the spectral troughs shallower.

Connection to material science

Hyperuniform-derived networks provide a flexible way to **design new metamaterials with tunable spectral properties**.

Random matrix perspective

Eigenvalues of random matrices

Eigenvalues of many random matrix ensembles form hyperuniform point processes

- **Sine process** (on the real line) eigenvalues of a random Hermitian matrix (e.g. the Gaussian Unitary Ensemble).
- **Ginibre ensemble** (in the plane) eigenvalues of a random matrix with independent complex Gaussian entries. Its structure factor is

$$S(k) = 1 - e^{-k^2/4}.$$

Takeaway

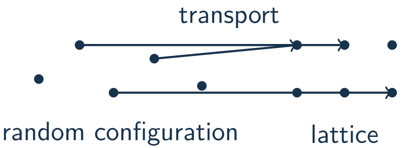
These processes have explicit correlation kernels, which allows one to compute

- number variance,
- structure factor,

Optimal transport perspective

Wasserstein distance

We would like to measure how far a point configuration is from a **perfectly uniform reference**, such as a lattice.



Wasserstein distance

The p-Wasserstein distance measures the minimal cost of transporting one configuration into another:

$$\mathrm{Wass}_p^p(\mathcal{P}, \mathcal{Q}) = \inf_C \int |x - y|^p dC(x, y).$$

Optimal transport and hyperuniformity

Optimal transport reveals a strong connection between **transport cost** and **fluctuation suppression**.

Finite transport cost implies hyperuniformity

Dereudre, Flimmel, Huesmann, Leblé (2025)

$$d = 1, 2: \qquad \mathrm{Wass}_d(\mathcal{P}, \text{lattice}) < \infty \quad \Rightarrow \quad \mathcal{P} \text{ is hyperuniform.}$$

Partial converse

Lachièze-Rey, Yogeshwaran (2024)

$$d = 2: \qquad \mathcal{P} \text{ hyperuniform} + \text{integrability} \quad \Rightarrow \quad \mathrm{Wass}_2(\mathcal{P}, \text{lattice}) < \infty.$$

Open problem: Theory missing for $d \geq 3$.

Big questions

Equivalent definitions?

Several notions are used to define hyperuniformity.

- **Number variance scaling**

  $$\mathrm{Var}[N_R] = o(R^d)$$

  for large observation windows (may depend on window shape).

- **Fourier formulation** vanishing structure factor

  $$S(k) \to 0 \quad \text{as } k \to 0,$$

  which requires sufficient regularity of correlations.

- **Pair correlation viewpoint** cancellation and decay properties of the two-point function.

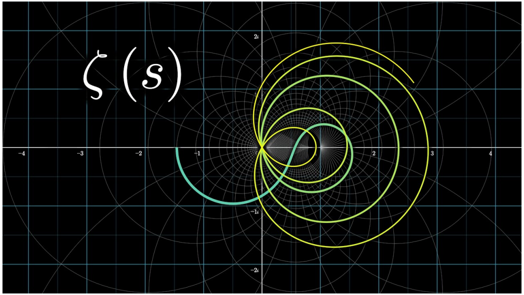
In full generality, the equivalence of these definitions remains open.

Riemann zeta function and hyperuniformity

Montgomery's pair correlation conjecture

After suitable rescaling, the non-trivial zeros of $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ have the same two-point correlation as bulk eigenvalues of the **Gaussian Unitary Ensemble (GUE)**

If true, the rescaled zeros would inherit the **hyperuniform behavior**.



Algorithmic generation of hyperuniform systems

A computational challenge

While many hyperuniform systems arise in physics and probability, we still lack **efficient algorithms** to construct them.

Algorithmic questions

- **Sampling:** Can we efficiently sample random hyperuniform point processes?
- **Local rules:** Can hyperuniformity emerge from simple local interactions or growth rules?
- **Recognition:** What is the computational complexity of deciding whether a configuration is hyperuniform?

We can go on...

Stability

- How stable is hyperuniformity under **perturbations, thinning,..**?
- Which operations preserve hyperuniform structure?

New settings

- What is the right notion of hyperuniformity in **non-Euclidean or combinatorial spaces**?

Dynamics

- Which stochastic or deterministic dynamics **converge to hyperuniform states**?
- Are there natural **self-organizing mechanisms** producing hyperuniformity?

Thank you!

Questions?