

# Convection diffusion problems with non-integrable initial data

Liviu Ignat

Institute of Mathematics of the Romanian Academy  
Joint work with F. Quiros (UAM, Spain)

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# Few words about diffusion problems

Heat equation:

$$\begin{cases} u_t - \Delta u = 0 & \text{in } \mathbb{R} \times (0, \infty), \\ u(0) = u_0. \end{cases}$$

For any  $u_0 \in L^1(\mathbb{R})$  the solution  $u \in C([0, \infty), L^1(\mathbb{R}))$  is given by:

$$u(t, x) = (G(t, \cdot) * u_0)(x)$$

where

$$G(t, x) = (4\pi t)^{-1/2} \exp\left(-\frac{|x|^2}{4t}\right)$$

Decay of solutions,  $1 \leq p \leq q \leq \infty$ :

$$\|u(t)\|_{L^q(\mathbb{R})} \lesssim t^{-\frac{1}{2}(\frac{1}{p} - \frac{1}{q})} \|u_0\|_{L^p(\mathbb{R})}$$



## Theorem

For any  $u_0 \in L^1(\mathbb{R})$  and  $p \geq 1$  we have

$$t^{\frac{1}{2}(1-\frac{1}{p})} \|u(t) - MG_t\|_{L^p} \rightarrow 0,$$

where  $M = \int u_0$ .

Proof:

$$(G_t * u_0)(x) - MG_t(x) = \frac{1}{(4\pi t)^{1/2}} \int_{\mathbb{R}^d} \left( \exp\left(-\frac{|x-y|^2}{4t}\right) - \exp\left(-\frac{|x|^2}{4t}\right) \right) u_0(y) dy$$

+ Taylor expansion with integral reminder, etc...

Remark 1: assuming  $u_0 \in L^1(1 + |x|)$  gives a better convergence result:  
extra  $t^{1/2}$

Remark 2: More terms can be obtained using decomposition on Dirac deltas  
 $u_0 = M\delta_0 + M_1\delta'_0 + \dots$  Duoandikoetxea + Zuazua, CRAS 92



# A transport term

Similarly, for

$$u_t - u_{xx} + au_x = 0, x \in \mathbb{R},$$

we get

$$u(t, x) \simeq MG_t(x - at)$$

Also, for Burgers' equation

$$u_t - u_{xx} + a(u^2)_x = 0, x \in \mathbb{R},$$

$$u(t, x) \simeq \frac{1}{\sqrt{t}} u_{a,M}\left(\frac{x}{\sqrt{t}}\right).$$

How you get the first term in the asymptotic expansion?



# Scaling Arguments $u_\lambda(t, x) = \lambda u(\lambda^2 t, \lambda x)$

## Why?

- Leaves invariant the equation :  $(u_\lambda)_t = (u_\lambda)_{xx}$ , keeps the mass
- $u(t) - MG_t \rightarrow 0$  in  $L^1(\mathbb{R}^N) \Leftrightarrow u_\lambda(1) \rightarrow f_M$  in  $L^1(\mathbb{R})$  as  $\lambda \rightarrow \infty$

## How it works (Four step method: Vazquez-Kamin)

- scaling - write the equation for  $u_\lambda$
- estimates and compactness of  $\{u_\lambda\}$
- passage to the limit  $u_\lambda \rightarrow U$ ,  $u_\lambda(1) \rightarrow U(1)$
- identification of the limit:  $U_t = U_{xx}$ ,  
 $U(0) = M\delta_0$ :  $\lim_{t \rightarrow 0} \int_{\mathbb{R}} U(t, x)\varphi(x)dx = M\varphi(0)$  **Uniqueness!!!!**



# Remarks on the method

- A good method for nonlinear problems  
 $u_t - \Delta u + (|u|^{q-1}u)_x = 0$ ,  $u_t + (-\Delta)^\alpha u + (|u|^{q-1}u)_x = 0$   
and even non-homogenous operators  $u_t + \mathcal{L}u = 0$  (A.Garriz, L.I.)
- It works for the first term in the asymptotic expansion
- No idea how to get more terms for  $u_t = Au$  if the kernel is not explicit. Duandikoetxea & Zuazua decomposition works well when you convolute some nice kernel with Dirac's deltas
- It work for graphs having at least one infinite edge (AMPA21 - Heat, NODEA22 -  $(|u|^{q-1}u)_x$ )
- There are some excelent web notes on the method by G. Karch



# Asymptotic expansion, The first term

$$u_t - u_{xx} + \partial_x(|u|^{q-1}u) = 0$$

$$t^{\max\{\frac{1}{2}, \frac{1}{q}\}(1-\frac{1}{p})} \|u(t) - U(t)\|_{L^p(\mathbb{R})} \rightarrow 0, \text{ as } t \rightarrow \infty$$

Escobedo-Vazquez-Zuazua, Carpio, etc...

- **Diffusive**  $q > 2$

$$U_t - U_{xx} = 0, U(0) = M\delta_0, \quad p \in [1, \infty]$$

- **Critical**  $q = 2$

$$U_t - U_{xx} + \partial_x(u^2) = 0, U(0) = M\delta_0$$

- **Convective**  $1 < q < 2$ ;  
 $0 < q < 1$ , Laurencot-98, Endal-Quiros-LI-2024,

$$U_t + \partial_x(|u|^{q-1}u) = 0, U(0) = M\delta_0, \quad p \in [1, \infty)$$



# Nonintegrable data

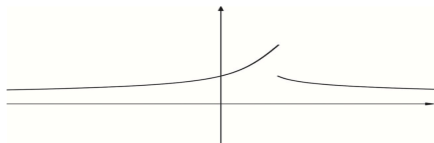
T.-E. Ghoul, N. Masmoudi and E. Pacherie, Nonlinear enhanced dissipation in viscous Burgers type equations, CPDE 2023, AIHP-C 2025

$$u_t - u_{xx} + (u^2)_x = 0$$

$$u_0(x) \simeq \frac{k_{\pm}}{|x|^{\gamma}}, \gamma \in (0, 1), x \rightarrow \pm\infty$$

Hopf-Cole transform

$$\lim_{t \rightarrow \infty} t^{\frac{\gamma}{\gamma+1}} u(t, t^{\frac{1}{1+\gamma}} z) \rightarrow p(z)$$



Q: is there any result for  $|u|^{q-1}u$ ,  $q \neq 2$ ?

Previous works for Porous Medium Equation

$$u_t = \Delta u^m, \quad u_t = \Delta u^m - u^p$$



N. D. Alikakos and R. Rostamian, On the uniformization of the solutions of the porous medium equation in  $\mathbf{R}^N$ , Israel J. Math. **47** (1984), no. 4, 270–290; MR0764297



S. Kamin and M. Ughi, On the behaviour as  $t \rightarrow \infty$  of the solutions of the Cauchy problem for certain nonlinear parabolic equations, J. Math. Anal. Appl. **128** (1987), no. 2, 456–469;

- $u_0(x) \simeq \frac{1}{|x|^\gamma}, \gamma \in (0, 1), x \rightarrow \pm\infty$   
 $u_0 \in L^p(\mathbb{R})$  for any  $p > 1/\gamma$  but all the results will not be optimal when using  $L^p - L^q$  decay of the linear semigroup
- $u_0(x) = \frac{1}{|x|^\gamma}, \gamma \in (0, 1)$ , even worse

$$u_0 \in L^{1/\gamma, \infty}(\mathbb{R})$$

The weak  $L^p$ -spaces or Marcinkiewicz spaces

$$\|f\|_{L^{p, \infty}(\mathbb{R})} = \sup_{t>0} t^{\frac{1}{p}} f^*(t) = \sup_{\lambda>0} \lambda |\{x \in \mathbb{R}, |f(x)| > \lambda\}|^{1/p}.$$

$$\begin{cases} u_t - u_{xx} + (|u|^{q-1}u)_x = 0, & x \in \mathbb{R}, \\ u(0) = u_0. \end{cases} \quad (1)$$

- $u_0 \in L^p(\mathbb{R})$  classical
- $u_0 \in L^{p_0,\infty}(\mathbb{R}) \cap L^\infty(\mathbb{R}) \subset L^p(\mathbb{R})$  for all  $p > p_0$

$$\|u(t)\|_{L^p(\mathbb{R})} \leq \|u_0\|_{L^p(\mathbb{R})}$$

- For any  $p_0 \in (1, \infty)$  there is a unique solution  $u \in C([0, \infty), L^{p_0,\infty}(\mathbb{R}))$

$$\|u(t)\|_{L^{p,\infty}(\mathbb{R})} \leq \left(\frac{C}{p-1}\right)^{1/p} \|u_0\|_{L^{p_0,\infty}(\mathbb{R})}, \quad \text{for all } p_0 \leq p < \infty.$$

- Previous work of Boccardo JEE2023, Parabolic estimates for

$$u_t - \operatorname{div}(a(t, x, \nabla u)) = 0$$

## Theorem

Let  $p_0 \in (1, \infty)$  and  $u_0 \in L^{p_0, \infty}(\mathbb{R}) \cap L^\infty(\mathbb{R})$ . Then the following holds:

$$\|u(t)\|_{L^\infty(\mathbb{R})} \leq C(p_0) \begin{cases} \|u_0\|_{L^{p_0, \infty}(\mathbb{R})} t^{-\frac{1}{2p_0}}, & q \geq p_0 + 1, \\ C(\|u_0\|_{L^{p_0, \infty}(\mathbb{R})}) t^{-\frac{q}{2(q-1)(p_0+1)} + 0}, & q \in (2, 1 + p_0), \\ \|u_0\|_{L^{p_0, \infty}(\mathbb{R})}^{\frac{p_0}{p_0+q-1}} t^{-\frac{1}{p_0+q-1}}, & q \in (1, 2]. \end{cases} \quad (2)$$

- First part/Parabolic follows from Boccardo's arguments
- If  $u_0 \in L^{1/\gamma, \infty}(\mathbb{R})$  the critical case is  $q = 1 + 1/\gamma$
- The expected hyperbolic decay should be  $t^{-\frac{1}{p_0+q-1}}$  for  $q < p_0 + 1$



# Hyperbolic decay

## Lemma (Oleinik type estimates)

Let  $q > 1$ . Any nonnegative solution  $u$  satisfies

$$(u^r)_x(t, x) \leq \begin{cases} \frac{r}{q(q-1)t} \|u_0\|_{L^\infty(\mathbb{R})}^{r-(q-1)}, & q-1 \leq r \leq 1, \\ C(q, r) t^{-\frac{1}{2} - \frac{1}{2} \frac{r}{q-1}}, & 0 < r < 1, q > r+1. \end{cases} \quad (3)$$

## Lemma (Magic lemma)

Let  $u \in C^1(\mathbb{R})$  nonnegative such that for some  $p \in [1, \infty)$  and  $r \in (0, 1]$

$$\|u\|_{L^{p,\infty}(\mathbb{R})} \leq A \quad \text{and} \quad (u^r)_x \leq B \quad \text{in } \mathbb{R} \quad (4)$$

for some constants  $A$  and  $B$ . Then,

$$\|u\|_{L^\infty(\mathbb{R})} \leq C(r, p) A^{\frac{p}{p+r}} B^{\frac{1}{p+r}}. \quad (5)$$

# Convergence in expandable sets

## Theorem

Let  $u_0 \in L^\infty(\mathbb{R})$  such that

$$\lim_{x \rightarrow \pm\infty} |x|^\gamma u_0(x) \rightarrow k_\pm$$

for some  $\gamma \in (0, 1)$ . Then, for all  $R > 0$  we have

$$\lim_{t \rightarrow \infty} t^{\gamma/2} \int_{|\xi| \leq R} \left| u(t, t^{1/2}\xi) - U_\gamma(\xi) \right| = 0 \quad \text{if } q \geq 1 + \frac{1}{\gamma},$$

$$\lim_{t \rightarrow \infty} t^{\frac{\gamma}{1+(q-1)\gamma}} \int_{|\xi| \leq R} \left| u(t, t^{\frac{1}{1+(q-1)\gamma}}\xi) - U_\gamma(\xi) \right| = 0 \quad \text{if } q \in (1, 2],$$

where  $U_\gamma$  is the profile of the **unique** (self-similar) solution with initial datum  $k_+ x^\gamma$  if  $x > 0$ ,  $k_- (-x^\gamma)$  if  $x < 0$  to the heat equation if  $\gamma > 1 + \frac{1}{\gamma}$ , the complete equation if  $q = 1 + \frac{1}{\gamma}$  or the pure conservation law if  $q \in (1, 2]$ .

## Few ideas of proof

$$|u_0(x)| \lesssim \frac{1}{1+|x|^\gamma} \text{ with } \gamma \in (0, 1)$$

$$\|u(t)\|_{L^\infty(\mathbb{R})} \leq C(p_0) \begin{cases} \|u_0\|_{L^{1/\gamma, \infty}(\mathbb{R})} t^{-\gamma/2}, & q \geq 1 + \frac{1}{\gamma}, \\ \left( \frac{\|u_0\|_{L^{1/\gamma, \infty}(\mathbb{R})}^{1/\gamma}}{t} \right)^{\frac{\gamma}{1+(q-1)\gamma}}, & q \in (1, 2] \end{cases}$$

The behaviour of the solutions when  $q \in (2, 1 + 1/\gamma]$  is still open. In the case of pure conservation laws (without diffusion), Dafermos proved that for any  $q > 1$  solutions decay as

$$\|u(t)\|_\infty \lesssim t^{-\frac{\gamma}{1+(q-1)\gamma}}.$$



**The case**  $\underline{q \in (1, 2]}$ . In this case we consider the following rescaled family

$$u_\lambda(t, x) = \lambda^\gamma u(\lambda^{1+(q-1)\gamma}t, \lambda x).$$

$$u_{\lambda,t} + (F(u_\lambda))_x = \lambda^{(q-1)\gamma-1} u_{\lambda,xx}.$$

Idea:  $u_\lambda(1) \rightarrow U_\gamma$  in  $L^1_{loc}(\mathbb{R})$  where  $U_\gamma$  is the entropy solution of the equation

$$U_t + (F(U))_x = 0, \quad \text{in } Q := (0, \infty) \times \mathbb{R}, \quad U(0, x) = \frac{k \operatorname{sgn}(x)}{|x|^\gamma}, \quad \text{on } \mathbb{R}. \quad (6)$$

Big Problem: the uniqueness of the limit profile  $U_\gamma$ .

- $L^1_{loc}((0, T) \times \mathbb{R})$  entropy solutions, Panov's papers
- $U_\gamma$  is not any entropy solution, it inherits some properties from  $u_\lambda$



# Kružkov entropy solution

$F : \mathbb{R} \rightarrow \mathbb{R}$  be locally Lipschitz and  $u_0 \in L^\infty(\mathbb{R}^N)$

$$\begin{cases} \partial_t u + \partial_x(F(u)) = 0, & (t, x) \in (0, T) \times \mathbb{R}, \\ u(0, x) = u_0(x), & x \in \mathbb{R}. \end{cases} \quad (7)$$

A function  $u \in L^\infty((0, T) \times \mathbb{R})$  is called a *Kružkov entropy solution* if for every  $k \in \mathbb{R}$  and every  $\varphi \in C_c^\infty((0, T) \times \mathbb{R}^N)$ ,  $\varphi \geq 0$ , one has

$$\int_0^T \int_{\mathbb{R}^N} \left( |u - k| \partial_t \varphi + \operatorname{sgn}(u - k) (f(u) - f(k)) \cdot \nabla_x \varphi \right) dx dt \geq 0. \quad (8)$$

$$\lim_{t \downarrow 0} \int_K |u(t, x) - u_0(x)| dx = 0. \quad (9)$$

Kružkov72: unique bounded entropy solution



# Our uniqueness result

## Theorem

Let  $\gamma \in (0, 1)$  and  $|F(u)| \lesssim |u|^q$  with  $1 < q < 1 + \frac{1}{\gamma}$ . For any function  $u_0$  satisfying

$$|u_0(x)| \lesssim \frac{1}{|x|^\gamma}, \quad \forall x \in \mathbb{R},$$

there exists at most one entropy solution  $U \in L^\infty((\tau, \infty) \times \mathbb{R})$  for all  $\tau > 0$  satisfying the decay property

$$|U(t, x)| \lesssim t^{-\frac{\gamma}{1+(q-1)\gamma}}, \quad \forall t > 0. \quad (10)$$

For  $1 < q \leq 2$  existence from the limit of  $u_\lambda$ .

For  $2 < q < 1 + 1/\gamma$  only uniqueness.



## Theorem

Let  $q > 1$  and

$$u_0(x) = \begin{cases} k_+/x^\gamma, & x > 0, \\ k_-/|x|^\gamma, & x < 0, \end{cases} \quad \gamma \in (0, 1), \quad k_+, k_- \in \mathbb{R} \setminus \{0\}. \quad (11)$$

For any  $k_+$  and  $k_-$  positive there exists a self-similar entropy solution in the uniqueness class (when  $q < 1 + \frac{1}{\gamma}$ )

$$U(t, x) = t^{-\alpha} V\left(\frac{x}{t^\beta}\right), \quad (12)$$

$$\alpha = \frac{\gamma}{1 + \gamma(q-1)}, \quad \beta = \frac{1}{1 + \gamma(q-1)},$$

and profile  $V \in L^\infty(\mathbb{R})$  satisfies  $\lim_{\xi \rightarrow \pm\infty} |\xi|^\gamma V(\xi) = k_\pm$ .

Moreover, there exists a positive constant  $\varepsilon_*(\gamma, q)$  such that the same result holds for  $k_- < 0 < k_+$  with  $k_+ < \varepsilon_*(\gamma, q)|k_-|$ .

## Asymptotic expansion again

- $\gamma = 1$  is special

$$|u_0(x)| \lesssim \frac{1}{(1+|x|)^\gamma}, \quad x \in \mathbb{R}.$$
$$|G_t * u_0|_{L^\infty(\mathbb{R})} \lesssim \begin{cases} (t+1)^{-\gamma/2}, & \gamma \in (0, 1), \\ (t+1)^{-1/2} \log(2+t), & \gamma = 1. \end{cases} \quad (13)$$

### Theorem

Let  $u_0 \in L^\infty(\mathbb{R})$  such that

$$|u_0(x)| \lesssim \frac{1}{1+|x|} \quad \text{for all } x \in \mathbb{R}. \quad (14)$$

Then

$$\|u(t)\|_{L^\infty(\mathbb{R})} \leq \begin{cases} (t+1)^{-1/2} \log(2+t), & q > 2, \\ (t+1)^{-1/q} \log^{1/q}(2+t), & 1 < q \leq 2. \end{cases} \quad (15)$$

## Theorem

Let  $k_{\pm} \in \mathbb{R}$  with  $k_+ + k_- \neq 0$  and  $u_0 \in L^\infty(\mathbb{R})$  be such that

$$\lim_{x \rightarrow \pm\infty} |x|u_0(x) \rightarrow k_{\pm}.$$

Then, for all  $R > 0$  we have

$$\lim_{t \rightarrow \infty} \sup_{|\xi| \leq R} \left| \frac{2t^{1/2}}{\log t} u(t, t^{1/2}\xi) - (k_+ + k_-)G_1(\xi) \right| = 0 \quad \text{if } q > 2, \quad (16)$$

$$\lim_{\tau \rightarrow \infty} \sup_{|\xi| \leq R} \left| g(\tau)u(\tau g^{q-1}(\tau), \tau\xi) - U(\xi) \right| = 0 \quad \text{if } q \in (1, 2], \quad (17)$$

where  $g(\tau) = \tau / \log \tau$  and  $U$  is the  $N$ -wave profile, i.e. the unique entropy solution of

$$u_t + (F(u))_x = 0 \quad \text{in } Q := (0, \infty) \times \mathbb{R}, \quad u(0) = (k_+ + k_-)\delta_0.$$

# To be continued...

- $2 < q < 1 + \frac{1}{\gamma}$  no idea how to extend/what to use instead of the Oleinik estimate
- $0 < q < 1$  and  $0 < \gamma \leq 1$ . For  $\gamma > 1$ , integrable data Laurençot AMPA 1998
- Fractional diffusion  $(-\Delta)^s$
- Nonlocal Diffusion  $u_t = J * u - u$
- Vorticity equation, some connection with Gallay works for Oseen vortices for integrable data
- With J. Rossi we have results for infinite edges graphs for integrable data AMPA21, NODEA22,  $u_t = \Delta_\Gamma u + (u^q)_x$ ,  $q \geq 2$ .



Thanks for your attention!