Parabolic Problems in Fractal Domains: Autonomous vs Non-Autonomous Results and Open Problems

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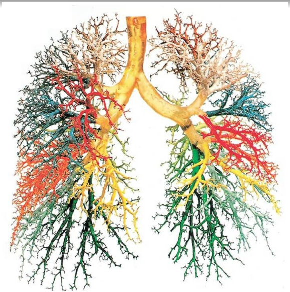
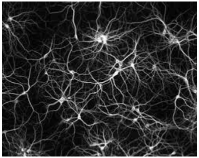
Outline

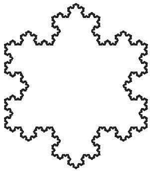
1. Fractals in nature
2. Motivations, why to study diffusion in fractal domains.
3. Tools: Functional spaces, Trace theorems on fractal domains, Integral theorems.
4. Autonomous problems in extension domains
5. The non autonomous case
6. Well posednness results
7. Current extensions and open problems

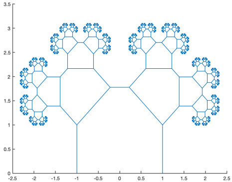
Fractals in nature

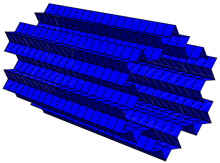
,









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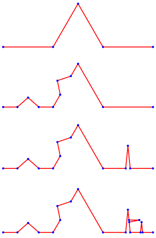
ξ_1=3
ξ_2=3.5
ξ_3=2.2
ξ_4=2.2

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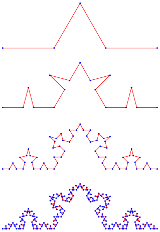
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Fractal boundaries/ layers $==>$ Large surfaces vs small volumes.

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Can we model diffusion theorethically or numerically?

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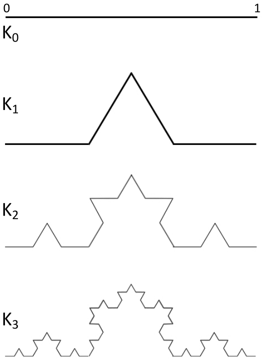
The Koch curve K



$$\Psi = \{\psi_i, i = 1, \dots, 4\}$$

$$K_{i_1 i_2 \dots i_n} = \psi_{i_1 i_2 \dots i_n}(K_0),$$

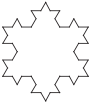
$$K_n = \bigcup_{i_1 i_2 \dots i_n = 1}^{4} K_{i_1 i_2 \dots i_n}$$

$$spt \mu_K = K,$$

$$\mu_K = (\mathcal{H}^{d_f}(K))^{-1} \mathcal{H}^{d_f}|_K, \text{ where } d_f = \frac{\ln 4}{\ln 3}$$

The SNOWFLAKE F AND THE FRACTAL PIPE $S = F \times I$, $I = [0,1]$

The snowflake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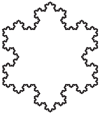
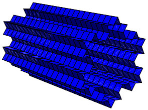


$F = \bigcup_{i=1,2,3} K_i = \bigcup_{i=4,5,6} K_i$
$spt\mu_F = F$,
$\mu_F = (\mathcal{H}^{d_f}(F))^{-1}\mathcal{H}^{d_f}|_F$,
where $d_f = \frac{\ln 4}{\ln 3}$

The Fractal Pipe



$S = F \times I$, $I = (0,1)$ $P \in S$,
$P = (x, y)$; $x = (x_1, x_2)$,
$x = (x_1, x_2) \in F$, $y \in I$.

$spt(g) = S$
$dg = d\mu_F \times dy$;
$\mathrm{d}y$ Lebesgue meas. on I
$\mathrm{d} = d_f + 1 = \frac{\log 12}{\log 3}$ S is a
d–set with $\mathrm{d} = d_f + 1$.

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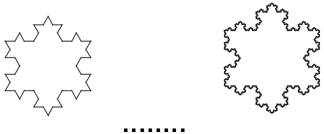

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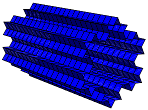

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We investigate problems

$$(P)\begin{cases} \partial_t^\alpha u - Au = f, & t \in (0,T]; \\ u(0) = u_0, \end{cases}$$

where $\alpha \in (0,1)$, A is a closed linear operator with domains $D(A) \subset Y$, Y Banach space, $f : [0,\infty) \to Y$, $u_0 \in Y$ are given.

If $\alpha = 1$

$$(P)\begin{cases} \partial_t u - Au = f, & t \in (0,T]; \\ u(0) = u_0, \end{cases}$$

A_n is the generator of a suitable energy form.
A, A_n can be a Laplacian, a fractional Laplacian, ... with various bcs (Robin, Venttsel)

The problems $(P), (P_n)$

We investigate problems

$$(P_n) \begin{cases} \partial_t^\alpha u_n - A_n u_n = f_n, & t \in (0,T]; \\ u_n(0) = u_0^n, \end{cases}$$

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Our approach

- Well posedness$\Rightarrow$ existence, uniqueness and regularity by introducing suitable energy forms in the irregular domain Ω.
- Approximate Ω in terms of suitable piecewise smooth domains Ω_n; $\Omega_n \to \Omega, n \to \infty$.
- Solve the corresponding approximating problems (P_n) : well posedness and regularity results.
- Asymptotics

$$P_n \to P, n \to \infty.$$

- Regularity $\Rightarrow$ numerical approximation , a priori error estimates ${u_n}^h \to u_n, h \to 0$.
- ${u_n}^h \to u, h \to 0, n \to \infty$. Difficult!

Our results

- Local Venttsel problems $\Rightarrow$ Asymptotics i.e Convergence of the solutions.
  - Divergence operators with drift terms: Hinz, L, Teplyaev, Vernole, JMAA 2018
  - Nondivergence operators L., Regis Durante, Vernole, DCDS-S 2016
  - Semilinear operators L., Vernole, NTMA 2012,2013

Our results

- Nonlocal Venttsel problems $\alpha = 1$
  - Fractional operators in space and/or time $\Rightarrow$ Convergence of the solutions.
    Creo, L., Vernole, JEE 2020, JCA 2021, Creo, L. NODEA 2021, DCDS-S 2022
  - Linear or quasilinear operators with nonlocal terms supported on the boundary.
    L., Velez-santiago,Vernole, NORWA 2017, Creo,Cefalo,L.,Vernole MMAS 2019, Creo, L, Nazarov, Vernole, DCDS 2019, Creo,L., Nazarov, FCAA 2019.

    $\Rightarrow$ Open problem: Convergence of the solutions when the nonlocal term is supported on the boundary!!
- Nonlocal Venttsel problems $\alpha \in (0, 1)$

  Capitanelli, Creo, L., Fractal Fract. 2023

Venttsel Problems in piecewise smooth or fractal domains

$$P \begin{cases} u_t(t,P) - Lu(t,P) = f(t,P) & \text{on } [0,T] \times Q, \\ u_t(t,P) - \eta_{\partial Q} L_{\partial Q} u(t,P) = & \\ -\frac{\partial u(t,P)}{\partial n} + f(t,P) & \text{on } [0,T] \times \partial Q, \\ u(0,P) = u_0 & \text{on } Q \end{cases}$$

$L, L_{\partial Q}$ are second order operators, $\eta_{\partial Q} \in \mathbb{R}^+$, Q is a bounded domain in $\mathbb{R}^2$.

e.g: $L = \Delta$, $L_{\partial Q}$ is $\Delta_{\partial Q}$ the Laplace-Beltrami Operator or the fractal Laplacian

The Venttsel energy :

$$E[u] = \int_Q |\nabla u|^2 d\mathcal{L}_2 + \eta_{\partial Q} E_{\partial Q}[u], u \in D(E) \subset L^2(Q, m).$$

- Linear Venttsel' BVPs + drift terms (Hinz, L., Teplyaev, Vernole JMAA 2018)
- Semilinear Venttsel' BVPs (L.Vernole NTMA 2012, AM 2014, IJPDE 2014)

$$\begin{cases} u_t(t,P) - \Delta u(t,P) = f(u(t,P)) & \text{in } [0,T]\times\Omega, \\ u_t(t,P) - \eta_S \Delta_S u(t,P) + b(P)u(t,P) = -\frac{\partial u(t,P)}{\partial n_{\mathcal{A}}} + f(u(t,P)) & \text{on } [0,T]\times K, \\ u(0,P) = 0 & \text{on } \Omega, \end{cases}$$

- Venttsel' BVPs of Ambrosetti-Prodi type (L., Velez-Santiago, Vernole DCDS 2019)

For $2D$ irregular domains Q: NONLOCAL QUASI LINEAR VENTTSEL PROBLEMS:

$$\begin{cases} c_Q u_t(t,P) - \eta_Q \Delta_p u(t,P) = f(t,P) & \text{on } [0,T]\times\Omega, \\ c_S u_t(t,P) - \eta_S \Delta_{S,p} u(t,P) = -\frac{\partial u(t,P)}{\partial n_p} + g(t,P) + \theta_p(u) & \text{on } [0,T]\times K, \\ u(0,P) = 0 & \text{on } \Omega \end{cases}$$

(Creo, Cefalo, L., Nazarov, Velez-Santiago, Vernole, (Norwa 2017, DCDS2019, Math. Meths. Appl. Sci. 2019))

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(Creo, Cefalo, L., Nazarov, Velez-Santiago, Vernole, (Norwa 2017, DCDS2019, Math. Meths. Appl. Sci. 2019))

Fractional Venttsel Diffusion in extension domains

The nonlocal *Robin-Venttsel' problem* is formally stated as follows:

$$(\tilde{P}) \begin{cases} \partial_t u(t,x) + (-\Delta_p)^s_\Omega u(t,x) = f(t,x) & \text{in } (0,T] \times \Omega, \\ \partial_t u(t,x) + \mathcal{N}_p^{p'(1-s)} u + b|u|^{p-2}u + p\Theta_{p,\gamma}(u) = g & \text{on } (0,T] \times \partial\Omega, \\ u(0,x) = u_0(x) & \text{in } \overline{\Omega}. \end{cases}$$

(Creo, L. ,Vernole, JEE 2020, JCA 2021, DCDS-S 2022, Creo-L. NoDea 2021,)

($s = 1$, L., Velez-Santiago, Vernole NORWA 2017; $s = 1, p = 2$, Creo, L., Vernole, Nazarov, DCDS-S 2019, Creo, L., Nazarov FCAA 2020)

Go to : Fractional p-Laplacian Go to : Nonlocal term

The non local Venttsel-type Energy, $p = 2$

$$E[u] = \frac{C_{N,s}}{2} \int\int_{\Omega\times\Omega} K(x,y) \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} \, \mathrm{d}\mathcal{L}_N(x)\mathrm{d}\mathcal{L}_N(y)$$
$$+ \int_{\partial\Omega} b(P)|u|_{\partial\Omega}|^2 \, \mathrm{d}\mu + \langle \Theta^t_\alpha(u|_{\partial\Omega}), u|_{\partial\Omega} \rangle$$

defined on the domain $D(E) = H^s(\Omega)$
$s > \frac{N-d}{2}$, $\alpha := s - \frac{N-d}{2}$

- $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $(B^{2,2}_\alpha(\partial\Omega))'$ and $B^{2,2}_\alpha(\partial\Omega)$ and $\alpha = s - \frac{N-d}{2}$.
- this nonlocal term on $\partial\Omega$ is equivalent to the seminorm of $B^{2,2}_\alpha(\partial\Omega)$;
- we point out that it can be regarded as a regional fractional Laplacian of order α on the boundary.

The Asymptotic behaviour of the solutions: via Mosco's convergence

$$E^n \to A^n \to T^n(t)$$

- $E^n \to E$, $n \to \infty$, in the sense of Mosco or /Mosco-Kuwae Shioya
- Mosco convergence implies convergence of Resolvents, Trotter Kato theorem implies strong convergence of semigroups $T^n(t)$ in a suitable sense.
- From the convergence of $T^n(t)$ we deduce the convergence of the solutions $u_n \to u$ in a suitable sense. *Go to M-convergence*

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Tools

- Setting: Extension domains $\rightarrow$ Functional spaces,
- Trace Theorems
- Fractional Green formula

Irregular domains

Let $\Omega \subset \mathbb{R}^N$ be a (ϵ, δ) domain with boundary a *d*-set. We suppose that Ω can be approximated by a sequence $\{\Omega_n\}$ of domains such that for every $n \in \mathbb{N}$:

Go to: Trace theorem

Irregular domains

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$$(\mathcal{H}) \begin{cases} \Omega_n \text{ is bounded and Lipschitz;} \\ \Omega_n \subseteq \Omega_{n+1}; \\ \Omega = \bigcup_{n=1}^{\infty} \Omega_n. \end{cases}$$

Remark: Ω is a $W^{s,p}$-extension domain (A. Jonsson, H. Wallin (1984))

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Remark: Ω is a $W^{s,p}$-extension domain (A. Jonsson, H. Wallin (1984))

Go to: Trace theorem

The non autonomous fractional semilinear case, $p = 2$.

$$\begin{cases} \partial_t u(t,x) + \mathcal{B}^{s,t}_{\Omega} u(t,x) = J(u(t,x)) & \text{in } [0,T] \times \Omega, \\ \partial_t u(t,x) + C_s \mathcal{N}^K_{2-2s} u(t,x) + b(t,x)u(t,x) + \Theta^t_{\alpha}(u(t,x)) = J(u(t,x)) & \text{on } [0,T] \times \partial\Omega, \\ u(0,x) = \phi(x) & \text{in } \overline{\Omega}. \end{cases}$$

Ω extension domain in $\mathbb{R}^n$, b, ϕ are given functions, T is positive number, $\mathcal{N}^K_{2-2s}$ is the fractional normal derivative to be suitably defined

Usual setting : $L^2(\Omega, m)$, $\mathrm{d}m = \mathrm{d}\mathcal{L}_n + \mathrm{d}\mu$, $\|u\|^2_{L^2(\Omega,m)} = \|u\|^2_{L^2(\Omega)} + \|u|_{\partial\Omega}\|^2_{L^2(\partial\Omega)}$.
Contraction argument in Banach spaces $X = [D(A), L^2(\Omega, m)]_\Theta$ or $X = D(A^\alpha)$,

$D(A) = ???$

Go to: strong intepretation

The Abstract Problem

$$(P) \qquad \begin{cases} \frac{\mathrm{d}u(t)}{\mathrm{d}t} = A(t)u(t) + J(u(t)), & 0 \leq t \leq T \\ u(0) = \phi \end{cases}$$

Here $A(t) : \mathcal{D}(A(t)) \subset L^2(Q) \to L^2(Q)$

$$E[t,u] = \frac{C_{N,s}}{2} \int\int_{\Omega\times\Omega} K(t,x,y) \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} \mathrm{d}\mathcal{L}_N(x)\mathrm{d}\mathcal{L}_N(y)$$
$$+ \int_{\partial\Omega} b(t,P)|u|_{\partial\Omega}|^2 \,\mathrm{d}\mu + \langle \Theta^t_\alpha(u|_{\partial\Omega}), u|_{\partial\Omega}\rangle$$

defined on the domain $D(E) = [0,T] \times H^s(\Omega)$
$s > \frac{N-d}{2}$, $\alpha := s - \frac{N-d}{2}$

The non local term Θ^t_α

We now introduce a bounded linear operator $\Theta^t_\alpha : B^{2,2}_\alpha(\partial\Omega) \to (B^{2,2}_\alpha(\partial\Omega))'$ defined by

$$\langle \Theta^t_\alpha(u), v\rangle := \int\int_{\partial\Omega\times\partial\Omega} \zeta(t,x,y)\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{d+2\alpha}}\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y),$$

- $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $(B^{2,2}_\alpha(\partial\Omega))'$ and $B^{2,2}_\alpha(\partial\Omega)$ and $\alpha = s - \frac{N-d}{2}$.
- From our hypotheses on ζ, this nonlocal term on $\partial\Omega$ is equivalent to the seminorm of $B^{2,2}_\alpha(\partial\Omega)$;
- we point out that, if $\zeta \equiv 1$, it can be regarded as a regional fractional Laplacian of order α on the boundary.

Let $s \in (0,1)$. For every fixed $t \in [0,T]$, we define the operator $\mathcal{B}^{s,t}_\Omega$ acting on $H^s(\Omega)$ in the following way:

$$\begin{aligned}\mathcal{B}^{s,t}_\Omega u(t,x) &= C_{N,s}\mathrm{P.V.}\int_\Omega K(t,x,y)\frac{u(t,x)-u(t,y)}{|x-y|^{N+2s}}\,\mathrm{d}\mathcal{L}_N(y)\\ &= C_{N,s}\lim_{\epsilon\to 0^+}\int_{\{y\in\Omega\,:\,|x-y|>\epsilon\}} K(t,x,y)\frac{u(t,x)-u(t,y)}{|x-y|^{N+2s}}\,\mathrm{d}\mathcal{L}_N(y).\end{aligned} \tag{1}$$

The positive constant $C_{N,s}$ is defined by

$$C_{N,s} = \frac{s2^{2s}\Gamma(\frac{N+2s}{2})}{\pi^{\frac{N}{2}}\Gamma(1-s)},$$

where Γ is the Euler function.

If $K \equiv 1$ on $[0,T]\times\Omega\times\Omega$, the operator $\mathcal{B}^{s,t}_\Omega$ reduces to the usual regional fractional Laplacian $(-\Delta)^s_\Omega$.

Properties of the energy form

Assuming simmetry, boundness, Hölder continuity in time for $K(t,x,y)$ and the kernel of Θ_α^t :

Proposition

For every $t \in [0,T]$, the form $E[t,u]$ is continuous and coercive on $H^s(\Omega)$, closed on $L^2(\Omega,m)$ and Markovian, (Dirichlet form).

Theorem

For every $u, v \in H^s(\Omega)$ and for every $t \in [0,T]$, $E(t,u,v)$ is a closed symmetric bilinear form on $L^2(\Omega,m)$. Then there exists a unique selfadjoint non-positive operator $A(t)$ on $L^2(\Omega,m)$ such that

$$E(t,u,v) = (-A(t)u, v)_{L^2(\Omega,m)} \quad \text{for every } u \in D(A(t)),\ v \in H^s(\Omega), \tag{2}$$

where $D(A(t)) \subset H^s(\Omega)$ is the domain of $A(t)$ and it is dense in $L^2(\Omega,m)$.

See Kato.
Go to: hypothesis on K-Theta-b:

Properties of the energy form

Proposition

For every $t \in [0, T]$, the form $E(t, u, v)$ has the square root property, i.e. $D(A(t))^{\frac{1}{2}} = H^s(\Omega)$. Moreover, there exists a constant $C > 0$ such that, for every $\eta \in (\frac{1}{2}, 1)$,

$$|E(t, u, v) - E(\tau, u, v)| \leq C|t - \tau|^{\eta} \|u\|_{H^s(\Omega)} \|v\|_{H^s(\Omega)}, \quad 0 \leq \tau, t \leq T. \tag{3}$$

Proposition

For every $t \in [0, T]$ and $\tau \geq 0$, $A(t)\colon D(A(t)) \to L^2(\Omega, m)$ is the generator of a semigroup $e^{\tau A(t)}$ on $L^2(\Omega, m)$ which is strongly continuous, contractive and analytic with angle $\omega_{A(t)} > 0$.

Proposition

For every $t \in [0, T]$, the operator $A(t)$ satisfies the following properties:

1) *the spectrum of $A(t)$ is contained in a sectorial open domain*

$$\sigma(A(t)) \subset \Sigma_\omega = \{\mu \in \mathbb{C} \,:\, |\mathrm{Arg}\,\mu| < \omega\}$$

for some fixed angle $0 < \omega < \frac{\pi}{2}$. The resolvent satisfies the estimate

$$\| (\mu - A(t))^{-1} \|_{\mathcal{L}(L^2(\Omega,m))} \leq \frac{M}{|\mu|}$$

for $M \geq 1$ independent from t and $\mu \notin \Sigma_\omega \cup 0$; moreover, $A(t)$ is invertible and $\|A(t)^{-1}\| \leq M_1$ with M_1 independent from t;

2) *$D(A(t)) \subset D(A(\tau))^{\frac{1}{2}} = H^s(\Omega), \quad 0 \leq \tau \leq t \leq T$; in particular, $D(A(t)) \subset D(A(\tau))^\nu$ for every ν such that $0 < \nu \leq \frac{1}{2}$;*

3) *$A(t)^{-1}$ is Hölder continuous in t in the sense of Yagi, i.e.,*

$$\left\| A(t)^{\frac{1}{2}} \left(A(t)^{-1} - A(\tau)^{-1} \right) \right\|_{\mathcal{L}(L^2(\Omega,m))} \leq C|t-\tau|^\eta, \tag{4}$$

with some fixed exponent $\eta \in \left(\frac{1}{2}, 1\right]$ and $C > 0$.

The family of evolution operators

Theorem

For every $t \in [0, T]$, let $A(t)\colon D(A(t)) \to L^2(\Omega, m)$ be the linear unbounded operator generator of the energy form $E(t, u, v)$. Then there exists a unique family of evolution operators $U(t, \tau) \in \mathcal{L}(L^2(\Omega, m))$ such that

1) $U(\tau, \tau) = \mathrm{Id}, \quad 0 \leq \tau \leq \mathrm{T}$;
2) $U(t, \tau)U(\tau, \sigma) = U(t, \sigma), \quad 0 \leq \sigma \leq \tau \leq t \leq T$;
3) *for every* $0 \leq \tau \leq t \leq T$ *one has*

$$\|U(t, \tau)\|_{\mathcal{L}(L^2(\Omega,m))} \leq 1, \tag{5}$$

and $U(t, \tau)$ *for* $0 \leq \tau \leq t < T$ *is a strongly contractive family on* $L^2(\Omega, m)$;

4) *the map* $t \mapsto U(t, \tau)$ *is differentiable in* $(\tau, T]$ *with values in* $\mathcal{L}(L^2(\Omega, m))$ *and* $\frac{\partial U(t,\tau)}{\partial t} = A(t)U(t, \tau)$;
5) $A(t)U(t, \tau)$ *is a* $\mathcal{L}(L^2(\Omega, m))$*-valued continuous function for* $0 \leq \tau < t \leq T$. *Moreover, there exists a constant* $C > 0$ *such that*

$$\|A(t)U(t, \tau)\|_{\mathcal{L}(L^2(\Omega,m))} \leq \frac{C}{t - \tau}, \quad 0 \leq \tau < t < T. \tag{6}$$

Theorem

Let $\Xi := \{(t,\tau) \in (0,T)^2 \,:\, \tau < t\}$*. For every* $p \in [1,+\infty]$ *there exists an operator* $U_p(t,\tau) \in \mathcal{L}(L^p(\Omega,m))$ *such that*

$$U_p(t,\tau)u_0 = U(t,\tau)u_0 \quad \text{for every } (t,\tau) \in \Xi\,, \text{ for every } u_0 \in L^p(\Omega,m) \cap L^2(\Omega,m).$$

Moreover, for every $\tau \geq 0$ *the map* $U_p(\cdot,\tau)$ *is strongly continuous from* (τ,∞) *to* $\mathcal{L}(L^p(\Omega,m))$ *for every* $t \geq \tau$ *and*

$$\|U_p(t,\tau)\|_{\mathcal{L}(L^p(\Omega,m))} \leq 1 \quad \text{for every } p \geq 1. \tag{7}$$

Ultracontractivity

We now prove the ultracontractivity of the evolution family $U(t,\tau)$.

Theorem

The evolution operator $U(t,\tau)$ is ultracontractive, i.e., for every $f \in L^1(\Omega, m)$ and $(t,\tau) \in \Xi$,

$$\|U_1(t,\tau)f(\tau)\|_{L^\infty(\Omega,m)} \leq \left(\frac{\lambda \bar{C}}{2\beta}\right)^{\frac{\lambda}{2}} (t-\tau)^{-\frac{\lambda}{2}} \|f(\tau)\|_{L^1(\Omega,m)}, \quad (8)$$

where we recall that $\lambda = \frac{2d}{d-N+2s}$, $\bar{C}$ is the positive constant depending on N, s, d and Ω appearing in Nash inequality and $\beta > 0$ is the coercivity constant of E.

Proof: Adapting the proof of Laasri and Mugnolo 2020 and Nash inequality Creo-L.2024

Nash inequality: L. Creo 2024

Proposition

Let $u \in H^s(\Omega)$. Then there exists a positive constant $\bar{C} = \bar{C}(N, s, d, \Omega)$ such that the following Nash inequality holds,

$$\|u\|_{L^2(\Omega,m)}^{2+\frac{4}{\lambda}} \leq \bar{C}\|u\|_{H^s(\Omega)}^2 \|u\|_{L^1(\Omega,m)}^{\frac{4}{\lambda}}, \tag{9}$$

where $\lambda = \frac{2d}{d-N+2s}$.

Proof: adapting the proof of Daners 2002.

The semilinear problems

$$(P)\begin{cases}\frac{\partial u(t)}{\partial t}=A(t)u(t)+J(u(t)) & \text{for } t\in[0,T],\\ u(0)=\phi,\end{cases} \tag{10}$$

$A(t)\colon D(A(t))\subset L^2(\Omega,m)\to L^2(\Omega,m)$ family of operators associated to $E[t,u], \phi\in L^2(\Omega,m)$.

HP. for every $t\in[0,T]$ $J: L^{2p}(\Omega,m)\to L^2(\Omega,m)$ for $p>1$, locally Lipschitz, i.e.,

$$\|J(u)-J(v)\|_{L^2(\Omega,m)}\leq l(r)\|u-v\|_{L^{2p}(\Omega,m)} \tag{11}$$

whenever $\|u\|_{L^{2p}(\Omega,m)}\leq r, \|v\|_{L^{2p}(\Omega,m)}\leq r$.

$$J(0)=0$$

(g) Let $a:=\frac{\lambda}{4}\left(1-\frac{1}{p}\right)$; there exists $0<b<a\,:\, l(r)=\mathcal{O}(r^{\frac{1-a}{b}}),\ r\to+\infty$,
λ is the constant appearing in Nash inequality.
Rem. $0<a<1$ for $N-2s\leq\frac{d}{2}$ and $p>1$.
Rem. If $J(u)=|u|^{p-1}u$, $l(r)=\mathcal{O}(r^{p-1})----->b=\frac{1}{p-1}-\frac{\lambda}{4p}, p>1+\frac{4}{\lambda}$.

Local existence theorem

Theorem

Let condition (g) *hold. Let* $\kappa > 0$ *be sufficiently small,* $\phi \in L^2(\Omega, m)$ *and*

$$\limsup_{t \to 0^+} \|t^b U(t,0)\phi\|_{L^{2p}(\Omega,m)} < \kappa. \tag{12}$$

Then there exists a $\overline{T} > 0$ *and a unique mild solution*

$$u \in C([0, \overline{T}], L^2(\Omega, m)) \cap C((0, \overline{T}], L^{2p}(\Omega, m)),$$

with $u(0) = \phi$ *and* $\|t^b u(t)\|_{L^{2p}(\Omega,m)} < 2\kappa$*, satisfying, for every* $t \in [0, \overline{T}]$*,*

$$u(t) = U(t,0)\phi + \int_0^t U(t,\tau) J(u(\tau))\, \mathrm{d}\tau, \tag{13}$$

with the integral being both an L^2*-valued and an* L^{2p}*-valued Bochner integral.*

proof: Inspired by Weissler 1980, L.Vernole 2012, 2022.

Crucial tools: Contraction argument in a space of continuous functions with values in Banach spaces , hypothesis (g), Lipschitz HP on J, mapping properties of $U(t, s)$ from L^2 to L^{2p}.

Regularity results

Theorem

Let the assumptions of the previous theorem hold.

a) *Let condition* (g) *hold. Then, the solution* $u(t)$ *can be continuously extended to a maximal interval* $(0, T_\phi)$ *as a solution of* (13), *until* $\|u(t)\|_{L^{2p}(\Omega,m)} < \infty$;

b)

$$u \in C([0, T_\phi), L^2(\Omega, m)) \cap C((0, T_\phi), L^{2p}(\Omega, m)) \cap C^1((0, T_\phi), L^2(\Omega, m)),$$

$$Au(t) \in \; C((0, T_\phi); \; L^2(\Omega, m))$$

and u satisfies

$$\frac{\partial u(t)}{\partial t} = A(t)u(t) + J(u(t)) \quad \textit{for every} \;\; t \in (0, T_\phi)$$

and $u(0) = \phi$ *(that is, u is a classical solution).*

Proof: Crucial tool $\mathcal{R}(U(t,\tau)) \subset D(A(t))$ for every $0 < \tau \leq t$, ultracontractivity of $U(t,\tau)$

Global existence theorem

Theorem

Let condition (g) *hold. Let* $q := \frac{2\lambda p}{\lambda+4pb}$, $\phi \in L^q(\Omega, m)$ *and* $\|\phi\|_{L^q(\Omega,m)}$ *be sufficiently small. Then there exists* $u \in C([0,\infty), L^q(\Omega, m))$ *which is a global solution of* (13).

Strong interpretation

In which sense,if any, the solution of problem (P)

$$(P) \begin{cases} u_t(t) = A(t)u(t) + J(u(t)) & \text{for } t \in [0, T], \\ u(0) = \phi, \end{cases}$$

solves the Nonlocal Venttsel semilinear problem?

Go to: Nonautonomous Venttsel Problem

Generalized Green formulas are a crucial tool.

Strong interpretation

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solves the Nonlocal Venttsel semilinear problem?

Go to: Nonautonomous Venttsel Problem

Generalized Green formulas are a crucial tool.

Green formula on Lipschitz domains

Definition

Let $\mathcal{T} \subset \mathbb{R}^N$ be a Lipschitz domain. Let
$u \in V(\mathcal{B}^{s,t}_{\mathcal{T}}, \mathcal{T}) := \{u \in H^s(\mathcal{T}) \, : \, \mathcal{B}^{s,t}_{\mathcal{T}} u \in L^2(\mathcal{T})$ in the sense of distributions$\}$.
We say that u has a weak fractional conormal derivative in $(H^{s-\frac{1}{2}}(\partial\mathcal{T}))'$ if there exists $g \in (H^{s-\frac{1}{2}}(\partial\mathcal{T}))'$ such that

$$\langle g, v|_{\partial\Omega}\rangle_{(H^{s-\frac{1}{2}}(\partial\mathcal{T}))', H^{s-\frac{1}{2}}(\mathcal{T})} = -\int_{\mathcal{T}} \mathcal{B}^{s,t}_{\mathcal{T}} u \, v \, \mathrm{d}\mathcal{L}_N \tag{14}$$

$$+ \frac{C_{N,s}}{2} \iint_{\mathcal{T}\times\mathcal{T}} K(t,x,y) \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}} \, \mathrm{d}\mathcal{L}_N(x) \mathrm{d}\mathcal{L}_N(y)$$

for every $v \in H^s(\mathcal{T})$. In this case, g is uniquely determined and we call $C_s \mathcal{N}_{2-2s} u := g$ the weak fractional conormal derivative of u, where

$$C_s := \frac{C_{1,s}}{2s(2s-1)} \int_0^\infty \frac{|z-1|^{1-2s} - (z \vee 1)^{1-2s}}{z^{2-2s}} \, \mathrm{d}z.$$

If $K(t,x,y) \equiv 1$ and $s \to 1^-$ in , we obtain the Green formula for Lipschitz domains see e.g Baiocchi Capelo .

Theorem (Generalized fractional Green formula)

There exists a bounded linear operator $\mathcal{N}_{2-2s}$ *from* $V(\mathcal{B}^{s,t}_{\mathcal{T}}, \Omega)$ *to* $(B^{2,2}_{\alpha}(\partial\Omega))'$. *The following generalized Green formula holds for every* $u \in V(\mathcal{B}^{s,t}_{\mathcal{T}}, \Omega)$ *and* $v \in H^s(\Omega)$,

$$C_s \left\langle \mathcal{N}_{2-2s} u, v|_{\partial\Omega} \right\rangle_{(B^{2,2}_{\alpha}(\partial\Omega))', B^{2,2}_{\alpha}(\partial\Omega)} = -\int_{\Omega} \mathcal{B}^{s,t}_{\Omega} u \, v \, \mathrm{d}\mathcal{L}_N + \frac{C_{N,s}}{2} \iint_{\Omega\times\Omega} K(t,x,y) \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}} \, \mathrm{d}\mathcal{L}_N(x) \mathrm{d}\mathcal{L}_N(y). \tag{15}$$

Tools: Limit arguments, Trace theorems

Strong interpretation

Theorem

Let $\alpha = s - \frac{n-d}{2}$ *and* $s \in (0,1)$ *be such that* $n-d < 2s < n$. *Let* u *be the unique solution of problem* (P). *Then for every fixed* $t \in (0,T]$*, one has*

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) + \mathcal{B}^{s,t}_{\Omega} u(t,x) = J(u(t,x)) & \text{for a.e. } x \in \Omega, \\ \frac{\partial u}{\partial t} + C_s \mathcal{N}_{2-2s} u + bu + \Theta^t_{\alpha}(u) = J(u) & \text{in } (B^{2,2}_{\alpha}(\partial\Omega))', \\ u(0,x) = \phi(x) & \text{in } L^2(\Omega, m). \end{cases} \tag{16}$$

References

Main Paper:

- S. Creo, M. R. Lancia, *Dynamic boundary conditions for time dependent fractional operators on extension domains*, Adv. Diff. Eq., 29 (2024).

Evolution Families & Non-autonomous Forms:

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Autonomous & Non-autonomous BVPs:

- Coclite, Goldstein, Goldstein (2009).
- Lancia, Vernole (NTMA 2012, AM 2014, IJPDE 2014, JEE 2022).
- Kato (1961), Monniaux & Rhandi (2000), Daners (2002), Ouhabaz (2005), Arendt & Monniaux (2016).

Time- Fractional Non Autonomous Problem: Formulation and Assumptions

We consider the non-autonomous time-fractional semilinear problem (P):

$$\begin{cases} \partial_t^\alpha u(t) = A(t)u(t) + J(u(t)) & t \in (0, T), \\ u(0) = u_0, \end{cases}$$

where ∂_t^α is the Caputo-type fractional derivative ($\alpha \in (0,1)$), and $-A(t)$ generates an analytic semigroup $T_t(\tau)$ on a Banach space X. *Go to derivatacaputo*

Acquistapace-Terreni (AT) Conditions

For $t \in [0, T]$ and $\omega \in (0, \pi/2)$ with $\sigma(A(t)) \subset \Sigma_\omega$:

1. **Time-invariant Domain:** $D(A(t)) = \mathbb{X}$ is dense in X and independent of t.
2. **Resolvent Estimate:** $\exists M \geq 1$ such that $\forall \lambda \notin \Sigma_\omega$,

$$\|(\lambda I - A(t))^{-1}\|_{\mathcal{L}(X)} \leq \frac{M}{1 + |\lambda|}.$$

3. **Hölder Continuity:** $\exists L > 0, \theta \in (0, 1]$ such that for $t, s, \tau \in [0, T]$,

Solution Operators and Mild Solution

To define the solution, we rely on the Wright type function $\Phi_\alpha(z)$ to bridge the fractional derivative and the analytic semigroup $T_t(\tau)$:

$$\phi_t(\tau) := \int_0^\infty \Phi_\alpha(z) T_t(\tau^\alpha z)\, \mathrm{d}z,$$

$$\psi_t(\tau) := \alpha\tau^{\alpha-1} \int_0^\infty z\Phi_\alpha(z) T_t(\tau^\alpha z)\, \mathrm{d}z.$$

The Solution Operators

We construct the non-autonomous solution operators $S_\alpha(t,\tau)$ and $P_\alpha(t,\tau)$ by solving suitable operator-valued Volterra integral equations:

$$S_\alpha(t,\tau) := \phi_\tau(t-\tau) + U(t,\tau), \quad P_\alpha(t,\tau) := \psi_\tau(t-\tau) + V(t,\tau).$$

This allows us to write the **mild solution** for the semilinear problem:

$$u(t) = S_\alpha(t,0)u_0 + \int_0^t P_\alpha(t,\tau) J(u(\tau))\, \mathrm{d}\tau.$$

Main Results: Ultracontractivity

Let $X = L^2(\Omega)$. We establish well-posedness without fractional powers of $A(t)$ or interpolation spaces, crucial when $D(A(t))$ is unknown.

Theorem (Ultracontractivity Estimates)

Assume that $T_t(\tau)$ is ultracontractive, meaning $\forall 1 \le p \le q \le \infty$:

$$\|T_t(\tau)\|_{\mathcal{L}(L^p(\Omega),L^q(\Omega))} \le C\tau^{-\lambda_A\left(\frac{1}{p}-\frac{1}{q}\right)}.$$

Then, for $0 \le \tau < t \le T$:

a) *If $\lambda_A(1/p - 1/q) < 1$, S_α is ultracontractive:*

$$\|S_\alpha(t,\tau)\|_{\mathcal{L}(L^p,L^q)} \le C_S(t-\tau)^{-\alpha\lambda_A\left(\frac{1}{p}-\frac{1}{q}\right)}.$$

b) *If $\lambda_A(1/p - 1/q) < 2$, P_α is ultracontractive:*

$$\|(t-\tau)^{1-\alpha}P_\alpha(t,\tau)\|_{\mathcal{L}(L^p,L^q)} \le C_P(t-\tau)^{-\alpha\lambda_A\left(\frac{1}{p}-\frac{1}{q}\right)}.$$

Consequence: These estimates allow a fixed-point argument in $C([0,T];X)$ to prove that the mild solution is **global in time**.

Application: The Fractional Heat Problem

Example: Fractional Diffusion with Time-Dependent Coefficients

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a **smooth boundary** (e.g., $\partial\Omega \in C^2$). We consider (P_H):

$$\begin{cases} \partial_t^\alpha u(t,x) = \nabla \cdot (a(t,x)\nabla u(t,x)) + J(u(t,x)) & \text{in } (0,T)\times\Omega, \\ u(t,x) = 0 & \text{on } (0,T)\times\partial\Omega, \\ u(0,x) = u_0(x) & \text{in } \Omega. \end{cases}$$

- **The Operator:** $A(t)u = \nabla \cdot (a(t,x)\nabla u)$. The matrix $a(t,x)$ is uniformly elliptic and Hölder continuous in time.
- **Regularity and AT Conditions:** To verify the Acquistapace-Terreni conditions, we need a time-independent domain. For Dirichlet conditions in $X = L^2(\Omega)$, we have $D(A(t)) = H^2(\Omega) \cap H_0^1(\Omega)$.
- **Role of the Smooth Boundary:** The $H^2(\Omega)$ spatial regularity, which is strictly required to bound the operators and integrate by parts, strongly relies on the smoothness of $\partial\Omega$.

Results

- Inverse problems:
  S.Creo, M.R.L., G. Mola, S.Romanelli, " Inverse problems in irregular domains: approximation via Mosco Convergence, Contemporary Mathematics 2026".
  S.Creo, M.R.L.,A.Mola, G. Mola, S.Romanelli, "Inverse fractional problems". to appear on IP 2026

Open problems

- Extension to the time fractional non autonomous problems to non constant domains operators $A(t)$
- Extension to the p-case
- Constructive approach: $P_n \to P$, $n \to \infty$
- More general operators...

Thank you for your attention

Basic definitions: Jones domains and d-sets

Definition

Let $\mathcal{F} \subset \mathbb{R}^N$ be open and connected. For $x \in \mathcal{F}$, let $d(x) := \inf_{y \in \mathcal{F}^c} |x - y|$. We say that $\mathcal{F}$ is an (ϵ, δ) domain if, whenever $x, y \in \mathcal{F}$ with $|x - y| < \delta$, there exists a rectifiable arc $\gamma \in \mathcal{F}$ joining x to y such that

$$\ell(\gamma) \leq \frac{1}{\epsilon}|x - y| \quad \text{and} \quad d(z) \geq \frac{\epsilon |x - z||y - z|}{|x - y|} \text{ for every } z \in \gamma.$$

Definition

A closed nonempty set $\mathcal{M} \subset \mathbb{R}^N$ is a d-set (for $0 < d \leq N$) if there exist a Borel measure μ with $\operatorname{supp} \mu = \mathcal{M}$ and two positive constants c_1 and c_2 such that

$$c_1 r^d \leq \mu(B(x, r) \cap \mathcal{M}) \leq c_2 r^d \quad \forall\, x \in \mathcal{M}.$$

The measure μ is called d-measure.

Basic definitions: Besov spaces

We recall the definition of Besov space specialized to our case.

Definition

Let $\mathcal{F}$ be a d-set with respect to a d-measure μ and $\alpha = s - \frac{N-d}{p}$.
$B^{p,p}_{\alpha}(\mathcal{F})$ is the space of functions for which the following norm is finite:

$$\|u\|^p_{B^{p,p}_{\alpha}(\mathcal{F})} = \|u\|^p_{L^p(\mathcal{G})} + \iint_{|x-y|<1} \frac{|u(x) - u(y)|^p}{|x - y|^{d+\alpha p}} \, \mathrm{d}\mu(x) \, \mathrm{d}\mu(y).$$

The trace Theorem

By $B^{p,p}_\alpha(S), 0 < \alpha < 1, 1 < p < \infty$ we denote the space of functions

$$B^{p,p}_\alpha(S) = \{u \in L^p(S) : \|u\|_{B^{p,p}_\alpha(S)} < +\infty\}$$

where

$$\|u\|_{B^{p,p}_\alpha(S)} = \|u\|_{L^p(S)} + \left(\int \int_{|x-y|\leq 1} \frac{|u(x) - u(y)|^p}{|x-y|^{\alpha p + d}} \mathrm{d}\mu(x)\mathrm{d}\mu(y) \right)^{\frac{1}{p}}$$

Theo. 1 Let $\alpha = s - \frac{n-d}{p}$, $s > \frac{n-d}{p}$, then $B^{p,p}_\alpha(S)$ is the trace space of $W^{s,p}(Q)$ in the following sense:

(i) γ_0 is a continuous linear operator from $W^{s,p}(Q)$ to $B^{p,p}_\alpha(S)$,

(ii) there is a continuous linear operator Ext from $B^{p,p}_\alpha(S)$ to $W^{s,p}(Q)$ such that $\gamma_0 \circ \mathrm{Ext}$ is the identity operator in $B^{p,p}_\alpha(S)$.

Go to: Irregular domains

The trace Theorem

By $B^{p,p}_{\alpha}(S), 0 < \alpha < 1, 1 < p < \infty$ we denote the space of functions

$$B^{p,p}_{\alpha}(S) = \{u \in L^p(S) : \|u\|_{B^{p,p}_{\alpha}(S)} < +\infty\}$$

where

$$\|u\|_{B^{p,p}_{\alpha}(S)} = \|u\|_{L^p(S)} + \left(\int \int_{|x-y|\leq 1} \frac{|u(x) - u(y)|^p}{|x - y|^{\alpha p + d}} \mathrm{d}\mu(x)\mathrm{d}\mu(y) \right)^{\frac{1}{p}}$$

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Go to: Irregular domains

- M. Warma, *NoDEA* (2016) and after (Lipschitz)

Let $s \in (0,1)$ and $p > 1$. For $\Gamma \subseteq \mathbb{R}^N$, we define the space

$$\mathcal{L}_s^{p-1}(\Gamma) := \left\{ u \colon \Gamma \to \mathbb{R} \text{ measurable } : \int_\Gamma \frac{|u(x)|^{p-1}}{(1+|x|)^{N+sp}} \, \mathrm{d}\mathcal{L}_N(x) < \infty \right\}.$$

The regional fractional p-Laplacian $(-\Delta_p)_\Gamma^s$ is defined as follows, for $x \in \Gamma$:

$$(-\Delta_p)_\Gamma^s u(x) = C_{N,p,s} \mathrm{P.V.} \int_\Gamma |u(x)-u(y)|^{p-2} \frac{u(x)-u(y)}{|x-y|^{N+sp}} \, \mathrm{d}\mathcal{L}_N(y)$$

$$= C_{N,p,s} \lim_{\varepsilon \to 0^+} \int_{\{y \in \Gamma \, : \, |x-y| > \varepsilon\}} |u(x)-u(y)|^{p-2} \frac{u(x)-u(y)}{|x-y|^{N+sp}} \, \mathrm{d}\mathcal{L}_N(y),$$

provided that the limit exists, for every function $u \in \Lambda_s^{p-1}(\Gamma)$. The positive constant $C_{N,p,s}$ is defined as follows:

$$C_{N,p,s} = \frac{s 2^{2s} \Gamma\left(\frac{ps+p+N-2}{2}\right)}{\pi^{\frac{N}{2}} \Gamma(1-s)},$$

where Γ is the Euler function.

Go to: Fractional diffusion

The Nonlocal term

We introduce the linear and continuous operator $\Theta_{p,\gamma}: B^{p,p}_{\alpha}(\partial\Omega) \to (B^{p,p}_{\alpha}(\partial\Omega))'$ defined as

$$\langle \Theta_{p,\gamma}(u), v \rangle :=$$

$$\frac{1}{p} \iint_{\partial\Omega\times\partial\Omega} \zeta(x,y) \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{d+\gamma p}} \, \mathrm{d}\mu(x)\, \mathrm{d}\mu(y),$$

where $\zeta \in L^{\infty}(\partial\Omega \times \partial\Omega)$ is such that $\zeta \geq 0$, $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $(B^{p,p}_{\alpha}(\partial\Omega))'$ and $B^{p,p}_{\alpha}(\partial\Omega)$ and $\gamma \in (0, \alpha]$, where $\alpha = s - \frac{N-d}{p} \in (0,1)$.

Go to: Fractional diffusion

Definition

Let $\mathcal{T} \subset \mathbb{R}^N$ be a Lipschitz domain. Let $u \in V((-\Delta_p)^s_{\mathcal{T}}, \mathcal{T}) := \{u \in W^{s,p}(\mathcal{T}) \, : \, (-\Delta_p)^s_{\mathcal{T}} u \in L^{p'}(\mathcal{T})$ in the sense of distributions$\}$. We say that u has a weak p-fractional normal derivative in $(W^{s-\frac{1}{p},p}(\partial\mathcal{T}))'$ if there exists $g \in (W^{s-\frac{1}{p},p}(\partial\mathcal{T}))'$ such that

$$\langle g, v|_{\partial\Omega} \rangle_{(W^{s-\frac{1}{p},p}(\partial\mathcal{T}))', W^{s-\frac{1}{p},p}(\mathcal{T})} = -\int_{\mathcal{T}} (-\Delta_p)^s_{\mathcal{T}} u \, v \, \mathrm{d}\Lambda_N$$

$$+ \frac{C_{N,p,s}}{2} \iint_{\mathcal{T}\times\mathcal{T}} |u(x) - u(y)|^{p-2} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+sp}} \, \mathrm{d}\Lambda_N(x) \mathrm{d}\Lambda_N(y)$$

for every $v \in W^{s,p}(\mathcal{T})$. In this case, g is uniquely determined and we call $C_{p,s}\mathcal{N}_p^{p'(1-s)} u := g$ the weak p-fractional normal derivative of u, where

$$C_{p,s} := \frac{(p-1)C_{1,p,s}}{(sp-(p-2))(sp-(p-2)-1)} \int_0^\infty \frac{|t-1|^{(p-2)+1-sp} - (t \vee 1)^{p-sp-1}}{t^{p-sp}} \, \mathrm{d}t.$$

Go to: Fractional diffusion

The p-fractional Green formula

We set $V((-\Delta_p)^s_\Omega, \Omega) := \{u \in W^{s,p}(\Omega) \, : \, (-\Delta_p)^s_\Omega u \in L^{p'}(\Omega)$ in the sense of distributions$\}$.

Theorem (p-fractional Green formula)

The following generalized Green formula holds for all $u \in V((-\Delta_p)^s_\Omega, \Omega)$ and $v \in W^{s,p}(\Omega)$:

$$C_{p,s} \left\langle \mathcal{N}_p^{p'(1-s)} u, v|_{\partial\Omega} \right\rangle_{(B^{p,p}_\alpha(\partial\Omega))', B^{p,p}_\alpha(\partial\Omega)} = - \int_\Omega (-\Delta_p)^s_\Omega u \, v \, \mathrm{d}\Lambda_N$$

$$+ \frac{C_{N,p,s}}{2} \iint_{\Omega\times\Omega} |u(x) - u(y)|^{p-2} \frac{(u(x) - u(y))(v(x) - v(y))}{|x-y|^{N+sp}} \, \mathrm{d}\Lambda_N(x) \mathrm{d}\Lambda_N(y).$$

The p-fractional Green formula

We set $V((-\Delta_p)^s_\Omega, \Omega) := \{u \in W^{s,p}(\Omega) : (-\Delta_p)^s_\Omega u \in L^{p'}(\Omega)$ in the sense of distributions$\}$.

Theorem (p-fractional Green formula)

The following generalized Green formula holds for all $u \in V((-\Delta_p)^s_\Omega, \Omega)$ and $v \in W^{s,p}(\Omega)$:

$$C_{p,s} \left\langle \mathcal{N}_p^{p'(1-s)} u, v|_{\partial\Omega} \right\rangle_{(B^{p,p}_\alpha(\partial\Omega))', B^{p,p}_\alpha(\partial\Omega)} = -\int_\Omega (-\Delta_p)^s_\Omega u \, v \, \mathrm{d}\Lambda_N$$

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Tools:

- p-fractional Green formula for Lipschitz domains
- properties of the approximating domains Ω_n

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Remark: when $s \to 1^-$, we recover the Green formula in (Lancia-Vernole, *Nonlinear Anal.* (2013)) for fractal domains.

THE CONVERGENCE OF HILBERT SPACES, (Kuwae-Shioya (2003))

Set

$$\{H_n\} = \{L^2(Q) \cap L^2(Q, m_n)\}, n \in \mathbb{N}, \qquad H = L^2(Q, m)$$

$\mathrm{d}m_n = \chi_{Q_n}\mathrm{d}\mathcal{L}_3 + \chi_{S_n}\delta_n\mathrm{d}\sigma$, $dm = d\Lambda_3 + dg$.

$$\|u\|^2_H = \|u\|^2_{L^2(Q)} + \|u\|^2_{L^2(S)}, \ \ \|u\|^2_{H_n} = \|u\|^2_{L^2(Q_n)} + \delta_n\|u\|^2_{L^2(S_n)}.$$

Theorem 10 Let $\delta_n = \left(\frac{3}{4}\right)^n$. Then the sequence $\{H_n\}_{n\in\mathbb{N}}$ converges to H in the sense of K-S..

Definition 11 The sequence of Hilbert spaces $\{H_n\}_{n\in\mathbb{N}}$ converges to the Hilbert space H if $\|u\|_{H_n} \to \|u\|_H, n \to \infty \qquad forany \quad u \in C(Q)$

Proposition 12 $\{u_n\} \in H_n$ converges to $u \in H$ iff
i) $\|u_n\|_{H_n} \to \|u\|_H, n \to \infty$
ii) $(u_n, \phi)_{H_n} \to (u, \phi)_H, \forall \phi \in C(Q)$

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The M-CONVERGENCE (U.Mosco 1969), Kuwae-Shioya (2003))

We extend the forms $E_s(\cdot,\cdot)$ and $E^{((n))_s}(\cdot,\cdot)$ on the spaces H and H_n respectively by defining

$$E_s(u,u) = +\infty \quad \text{for every } u \in L^2(Q,m) \setminus \Delta(E_s)$$

$$E^{((n))_s}(u,u) = +\infty \quad \text{for every } u \in H_n \setminus \Delta(E_s^{(n)}).$$

Definition 13

a) for every $\{v_n\} \in H_n$ weakly converging to $u \in H$ in $\mathcal{H}$

$$\underline{\lim}_{n\to\infty} E_s^{(n)}[v_n] \geq E_s[u];$$

b) for every $u \in H$ there exists a sequence $\{w_n\}$, with $w_n \in H_n$ strongly converging to u in $\mathcal{H}$, such that

$$\overline{\lim}_{n\to\infty} E_s^{(n)}[w_n] \leq E_s[u].$$

Go to: Tools

hypothesis on $b, K(t,x,y), \Theta_\alpha^t$

$$\begin{cases} b \in L^\infty([0,T]\times S), \\ \inf b(t,P) > b_0 > 0, \forall (t,P) \in [0,T]\times S, \\ \exists \eta \in (\frac{1}{2},1) : |b(t,P) - b(s,P)| \leq c|t-s|^\eta, \forall P \in S, \end{cases}$$

- $K(t,\cdot,\cdot) : [0,T]\times\Omega\times\Omega$ measurable fnc, symmetric for every $t \in [0,T]$: $\exists 0 < k_1 < k_2 : k_1 < K(t,x,y) < k_2, a.e. t \in t \in [0,T]$,
- ζ: $[0,T]\times\partial\Omega\times\partial\Omega \to \mathbb{R}$, symmetric for every fixed $t \in [0,T]$: $\exists 0 < \zeta_1 < \zeta_2 : \zeta_1 \leq \zeta(t,x,y) \leq \zeta_2$ for a.e $(t,x,y) \in [0,T]\times\partial\Omega\times\partial\Omega$.
- $\exists\, \eta \in (\frac{1}{2},1) : |K(t,x,y) - K(\tau,x,y)| \leq C|t-\tau|^\eta$, $\zeta(t,x,y) - \zeta(\tau,x,y)| \leq C|t-\tau|^\eta$

Go to: Energy-form

Let $\alpha \in (0,1)$. We define

$$g_\alpha(t) = \begin{cases} \dfrac{t^{\alpha-1}}{\Gamma(\alpha)} & \text{if } t > 0, \\ 0 & \text{if } t \leq 0, \end{cases}$$

where Γ is the usual Gamma function.

Definition (C. Gal, M. Warma (2020))

The Caputo-type fractional derivative of order $\alpha \in (0,1)$ is defined as follows:

$$\partial_t^\alpha f(t) := \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t g_{1-\alpha}(t-\tau)(f(\tau) - f(0))\, \mathrm{d}\tau,$$

for a.e. $t \in (0, T]$.

Go to Caputo