Double Nonlinear Diffusion Equations on Metric Graphs

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joint works with J. Toledo

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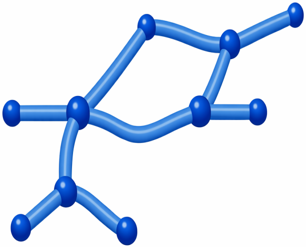
Introduction.



Tubular Network

Introduction. The Euclidean case

In the Euclidean space $\mathbb{R}^N$, the doubly nonlinear diffusion equation for the p-Laplacian,

$$\frac{\partial v}{\partial t} - \Delta_p u = f, \quad v \in \gamma(u), \tag{1}$$

with boundary conditions and initial data, where γ is a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$, $f(t,x)$ is a source, and

$$\Delta_p u := \mathrm{div}(|\nabla u|^{p-2}\nabla u), \quad 1 < p < \infty,$$

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For $m = 1$ and $p \in (1, \infty)$ the equation is known as the p-Laplacian evolution equation, and for $m \in (0, \infty)$ and $p = 2$ we are dealing with the porous medium equation or fast diffusion equation .

Preliminaires. Metric Graphs

A graph G consists of a finite or countable infinite set of vertices $V(G) = \{v_i\}$ and a set of edges $E(G) = \{e_j\}$ connecting the vertices. A graph G is said to be a finite graph if the number of edges and the number of vertices are finite.

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We define $\mathrm{E}_\mathrm{v}(\mathrm{G})$ as the set of all edges incident to v, and the degree of v as $d_\mathrm{v} := \sharp \mathrm{E}_\mathrm{v}(\mathrm{G})$. We define the boundary of V(G) as

$$\partial \mathrm{V}(\mathrm{G}) := \{\mathrm{v} \in \mathrm{V}(\mathrm{G}) \ : \ d_\mathrm{v} = 1\},$$

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We define $\mathrm{E}_\mathrm{v}(\mathrm{G})$ as the set of all edges incident to v, and the degree of v as $d_\mathrm{v} := \sharp \mathrm{E}_\mathrm{v}(\mathrm{G})$. We define the boundary of V(G) as

$$\partial \mathrm{V(G)} := \{\mathrm{v} \in \mathrm{V(G)} \ : \ d_\mathrm{v} = 1\},$$

We say that a graph G is a metric graph if

1. each edge e is assigned with a positive length $\ell_\mathrm{e} \in (0, +\infty]$;
2. for each edge e, a coordinate is assigned to each point of it, including its vertices. For that purpose, each edge e is identified with an ordered pair $(\mathrm{i}_\mathrm{e}, \mathrm{f}_\mathrm{e})$ of vertices, being i_e and f_e the initial and terminal vertex of e respectively and an increasing coordinate in the edge is defined in the direction of the edge:

$$\begin{array}{cccc} c_\mathrm{e}: & \mathrm{e} & \rightarrow & [0, \ell_\mathrm{e}] \\ & x & \rightsquigarrow & c_\mathrm{e}(x) \end{array}$$

such that $c_\mathrm{e}(\mathrm{i}_\mathrm{e}) = 0$, $c_\mathrm{e}(\mathrm{f}_\mathrm{e}) = \ell_\mathrm{e}$, and it is exhaustive.

Preliminaires. Metric Graphs

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A function u on a metric graph G is a collection of functions $[u]_{\mathrm{e}}$ defined on $(0, \ell_{\mathrm{e}})$ for all $\mathrm{e} \in \mathrm{E}(\mathrm{G})$, not just at the vertices as in discrete models.
We use the notation(we use the same name for variables in G and in $(0, \ell_{\mathrm{e}})$ since there is not ambiguity):

$$\int_{\mathrm{G}} u = \int_{\mathrm{G}} u(x)\,\mathrm{d}x := \sum_{\mathrm{e}\in\mathrm{E}(\mathrm{G})} \int_0^{\ell_{\mathrm{e}}} [u]_{\mathrm{e}}(x)\,\mathrm{d}x = \sum_{\mathrm{e}\in\mathrm{E}(\mathrm{G})} \int_0^{\ell_{\mathrm{e}}} [u]_{\mathrm{e}}.$$

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Let, for each $\mathrm{e} \in \mathrm{E(G)}$, $p_{\mathrm{e}} \in [1, +\infty]$. We write $\overline{p}$ to represent the collection of $(p_{\mathrm{e}})_{\mathrm{e}\in\mathrm{E(G)}}$. We denote by $L^{\overline{p}}(\mathrm{G})$ the space of all function in G such that $[u]_{\mathrm{e}} \in L^{p_{\mathrm{e}}}(0, \ell_{\mathrm{e}})$, for all $\mathrm{e} \in \mathrm{E(G)}$ with the norm

$$\|u\|_{\overline{p}} = \sum_{\mathrm{e}\in\mathrm{E(G)}} \|[u]_{\mathrm{e}}\|_{L^{p_{\mathrm{e}}}(0,\ell_{\mathrm{e}})}.$$

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Consider $\widetilde{\mathcal{W}}^{1,\overline{p}}(\mathrm{G})$ the space of functions in G such that $[u]_{\mathrm{e}} \in W^{1,p_{\mathrm{e}}}(0,\ell_{\mathrm{e}})$ for all $\mathrm{e} \in \mathrm{E}(\mathrm{G})$ with the Sobolev norm

$$\|u\|_{\widetilde{\mathcal{W}}^{1,\overline{p}}(\mathrm{G})} := \sum_{\mathrm{e}\in \mathrm{E}(\mathrm{G})} \left(\|[u]_{\mathrm{e}}\|^{p_{\mathrm{e}}}_{L^{p_{\mathrm{e}}}(0,\ell_{\mathrm{e}})} + \|[u]'_{\mathrm{e}}\|^{p_{\mathrm{e}}}_{L^{p_{\mathrm{e}}}(0,\ell_{\mathrm{e}})} \right)^{1/p_{\mathrm{e}}}.$$

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If $u \in \widetilde{W}^{1,\overline{p}}(\mathrm{G})$ we denote by u' the function in $L^{\overline{p}}(\mathrm{G})$ with $[u']_{\mathrm{e}} = [u]'_{\mathrm{e}}$ for all $\mathrm{e} \in \mathrm{E}(\mathrm{G})$.

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We denote by $C(\mathrm{V}(\mathrm{G}))$ the set of all functions u in G such that each $[u]_{\mathrm{e}}$ is continuous at $[0, \ell_{\mathrm{e}}]$ and

$$[u]_{\mathrm{e}_1}(\mathrm{v}) = [u]_{\mathrm{e}_2}(\mathrm{v}) \quad \text{for all } \mathrm{e}_1, \mathrm{e}_2 \in \mathrm{E}_{\mathrm{v}}(\mathrm{G}).$$

We denote this common value at v as $u(\mathrm{v})$.

Preliminaires. Metric Graphs

Observe that in the definition of $\widetilde{W}^{1,\overline{p}}(\mathrm{G})$ we do not assume the continuity at the vertices. Set

$$\mathcal{W}^{1,\overline{p}}(\mathrm{G}) = \widetilde{\mathcal{W}}^{1,\overline{p}}(\mathrm{G}) \cap C(\mathrm{V}(\mathrm{G})).$$

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If $I =]a,b[$ is a bounded interval in $\mathbb{R}$, we have that

$$\text{the injection } W^{1,p}(I) \subset C(\overline{I}) \text{ is compact for all } 1 < p \leq \infty. \tag{3}$$

Consequently we have the following result.

Theorem

Let G *be a compact metric graph and* $1 < p_{\mathrm{e}} < \infty$ *for all* $\mathrm{e} \in E(\mathrm{G})$*. The injection* $W^{1,\overline{p}}(\mathrm{G}) \subset C(\mathrm{G})$ *is compact.*

Preliminaires. Metric Graphs

Moreover, we have the following Poincaré inequality.

Theorem

Let G *be a compact metric graph and* $1 < p_{\mathsf{e}} < \infty$ *for all* $\mathsf{e} \in E(\mathsf{G})$. *There exists a constant* $\lambda_{\mathsf{G}} > 0$ *such that*

$$\lambda_{\mathsf{G}} \|u\|_{L^{\overline{p}}(\mathsf{G})} \leq \|u'\|_{L^{\overline{p}}(\mathsf{G})} \quad \forall\, u \in W^{1,\overline{p}}(\mathsf{G}) : \int_{\mathsf{G}} u = 0. \tag{4}$$

The Problem

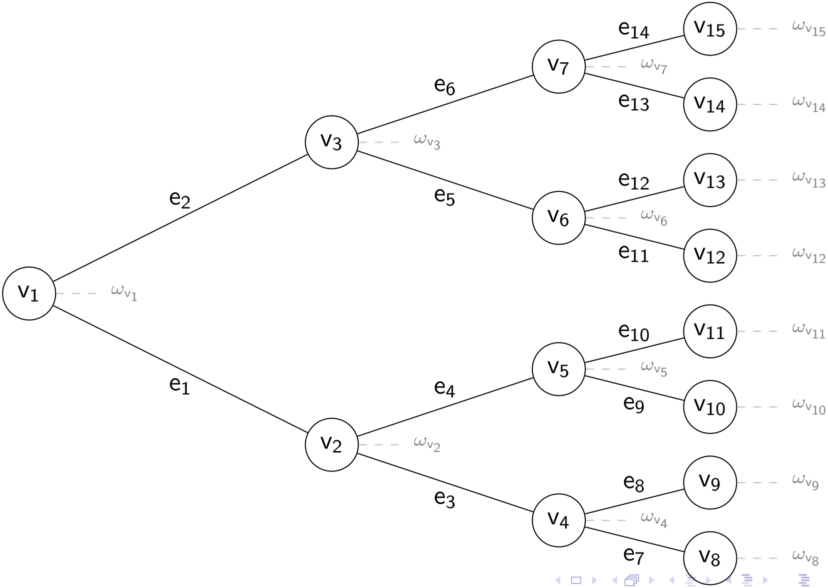
$$\begin{cases} \frac{\partial [v]_{\mathrm{e}}}{\partial t} - \Delta_{p_{\mathrm{e}}} [u]_{\mathrm{e}} = [f]_{\mathrm{e}} \quad \text{in } (0,+\infty) \times (0, \ell_{\mathrm{e}}), \ \forall\, \mathrm{e} \in \mathrm{E(G)}, \\ [v]_{\mathrm{e}} = \gamma_{\mathrm{e}}([u]_{\mathrm{e}}) \quad \text{in } (0,\infty) \times (0, \ell_{\mathrm{e}}), \ \forall\, \mathrm{e} \in \mathrm{E(G)}, \\ u \text{ continuous}, \\ \partial_\nu^{\overline{p}} u(t, \mathrm{v}) = \omega(t, \mathrm{v}) \quad \text{for all } (t, \mathrm{v}) \in (0,+\infty) \times \mathrm{V(G)}, \\ v(0) = v_0. \end{cases} \tag{5}$$

where G is a metric graph with vertices V(G) and edges E(G), $(0, \ell_{\mathrm{e}})$ is the interval where the edge e is parametrized, $\Delta_{p_{\mathrm{e}}}$ is the p_{e}-Laplacian operator, $\gamma_{\mathrm{e}} : \mathbb{R} \to \mathbb{R}$ is a continuous increasing function with $\gamma_{\mathrm{e}}(\mathbb{R}) = \mathbb{R}$ and $\gamma_{\mathrm{e}}(0) = 0$, $f(t,x)$ is a source term, and

$$\partial_\nu^{\overline{p}} u(t, \mathrm{v}) = \omega(t, \mathrm{v}) (= \omega_{\mathrm{v}}(t))$$

determines the flux trough a vertex v

The main result



The Problem

An important particular case of Problem (5) is when $p_{\mathrm{e}} = p$ and $\gamma_{\mathrm{e}} = \gamma$ for all $\mathrm{e} \in \mathrm{E(G)}$, that is, the problem

$$\begin{cases} \frac{\partial(\gamma([u]_{\mathrm{e}})}{\partial t} - \Delta_p [u]_{\mathrm{e}} = [f]_{\mathrm{e}} \quad \text{in } (0,+\infty) \times (0,\ell_{\mathrm{e}}), \ \forall\, \mathrm{e} \in \mathrm{E(G)}, \\ \\ \partial^p_\nu u(t,\mathrm{v}) = \omega_{\mathrm{v}} \quad \text{for all } (t,\mathrm{v}) \in (0,+\infty) \times \mathrm{V(G)}, \\ \\ v(0) = v_0, \end{cases} \tag{6}$$

In this case, if $\gamma(r) = r$ for all $r \in \mathbb{R}$, we have a pure p-Laplacian evolution problem in a metric graph; and for $p = 2$ and $\gamma(r) = |r|^{m-1}r$, with $m > 0$, we have a pure porous medium or a fast diffusion equation

Preliminaires. Results on a bounded interval

Given $\gamma : \mathbb{R} \to \mathbb{R}$ continuous and increasing with $\gamma(\mathbb{R}) = \mathbb{R}$ and $\gamma(0) = 0$, $1 < p < \infty$, $f \in L^1(0, \ell)$ and $a, b \in \mathbb{R}$, consider the problem

$$(P^{\gamma,f}_{p,a,b}) \quad \begin{cases} v - \Delta_p u = f \quad \text{in } (0, \ell), \\ v = \gamma(u) \quad \text{a.e. in } (0, \ell), \\ -\left(|u'|^{p-2}u'\right)(0) = a, \\ \left(|u'|^{p-2}u'\right)(\ell) = b. \end{cases}$$

Preliminaires. Results on a bounded interval

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Definition

We say that $v \in L^1(0,\ell)$ is a weak solution of problem $(P^{\gamma,f}_{p,a,b})$ if there exists $u \in W^{1,p}(0,\ell)$, with $v = \gamma(u)$ a.e. in $(0,\ell)$, such that

$$\int_0^\ell v\phi + \int_0^\ell |u'|^{p-2}u'\phi' = \int_0^\ell f\phi + b\phi(\ell) + a\phi(0) \quad \forall\, \phi \in W^{1,p}(0,\ell). \quad (7)$$

Preliminaires. Results on a bounded interval

If v is a weak solution to the problem $(P^{\gamma,f}_{p,a,b})$, we have the following mass balance property,

$$\int_0^\ell v = \int_0^\ell f + b + a, \tag{8}$$

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If v is a weak solution to the problem $(P_{p,a,b}^{\gamma,f})$, we have the following mass balance property,

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Theorem

Let $\gamma : \mathbb{R} \to \mathbb{R}$ be continuous and increasing with $\gamma(\mathbb{R}) = \mathbb{R}$ and $\gamma(0) = 0$, and $1 < p < \infty$. For $f \in L^1(0, \ell)$ and $a, b \in \mathbb{R}$, there exists a unique weak solution to problem $(P_{p,a,b}^{\gamma,f})$.

The elliptic problem on a metric graph.

Let G be a connected and compact metric graph (without loops and without multiple edges). For each $e \in E(G)$, let $\gamma_e : \mathbb{R} \to \mathbb{R}$ be continuous and increasing with $\gamma_e(\mathbb{R}) = \mathbb{R}$ and $\gamma_e(0) = 0$. We write $\overline{\gamma}$ to represent the collection of $(\gamma_e)_{e \in E(G)}$ and we write $[v]_e = \gamma_e([u]_e)$ for all $e \in E(G)$ as

$$v = \overline{\gamma}(u).$$

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We define the upwind values of a general function $z : \mathrm{G} \to \mathbb{R}$, interpreted as fluxes on G at vertices $\mathrm{v} \in \mathrm{V(G)}$, as follows. Let $\mathrm{v} \in \mathrm{V(G)}$ and $\mathrm{e} \in \mathrm{E_v(G)}$ be, we define

$$\{z\}_{\mathrm{e}}(\mathrm{v}) := \begin{cases} +[z]_{\mathrm{e}}(\ell_{\mathrm{e}}), & \text{if } \mathrm{v} = \mathrm{f_e}, \\ -[z]_{\mathrm{e}}(0), & \text{if } \mathrm{v} = \mathrm{i_e}, \end{cases} \tag{9}$$

whenever these values exist in the sense of traces, e.g., if $z \in \widetilde{W}^{1,1}(\mathrm{G})$.

The elliptic problem on a metric graph.

We have the the following Green's formula on a metric graph:

$$\int_{\mathrm{G}} z' w \, dx = - \int_{\mathrm{G}} z w' \, dx + \sum_{\mathrm{v} \in \mathrm{V}(\mathrm{G})} \left(\sum_{\mathrm{e} \in \mathrm{E}_{\mathrm{v}}(\mathrm{G})} \{z\}_{\mathrm{e}}(\mathrm{v}) \right) w(\mathrm{v}) \quad \text{for } z, w \in W^{1,1}(\mathrm{G}).$$

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Let $f \in L^1(\mathrm{G})$ be, and $\omega = \{\omega_{\mathrm{v}} : \mathrm{v} \in \mathrm{V}(\mathrm{G})\} \subset \mathbb{R}$, our aim in this section is to solve the elliptic problem

$$(P_\omega^f) \quad \begin{cases} [v]_{\mathrm{e}} - \Delta_{p_{\mathrm{e}}} [u]_{\mathrm{e}} = [f]_{\mathrm{e}} \quad \text{in } (0, \ell_{\mathrm{e}}), \ \forall \, \mathrm{e} \in \mathrm{E}(\mathrm{G}), \\ [v]_{\mathrm{e}} = \gamma_{\mathrm{e}}([u]_{\mathrm{e}}) \quad \text{a.e.}, \ \forall \, \mathrm{e} \in \mathrm{E}(\mathrm{G}), \\ u \text{ continuous}, \\ \partial_\nu^{\overline{p}} u(\mathrm{v}) = \omega_{\mathrm{v}} \quad \text{for all } \mathrm{v} \in \mathrm{V}(\mathrm{G}). \end{cases} \tag{10}$$

The elliptic problem on a metric graph.

Here,

$$\partial_{\nu}^{\overline{p}} u(\mathrm{v}) := \sum_{\mathrm{e} \in \mathrm{E_v(G)}} \{z_{\overline{p}}\}_{\mathrm{e}}(\mathrm{v}), \tag{11}$$

being $z_{\overline{p}}$ defined as

$$[z_{\overline{p}}]_{\mathrm{e}} = |[u]_{\mathrm{e}}'|^{p_{\mathrm{e}}-2}[u]_{\mathrm{e}}',$$

so that,

$$\partial_{\nu}^{\overline{p}} u(\mathrm{v}) = \omega_{\mathrm{v}}$$

is a generalized Neumann-Kirchhoff flux condition, which of course includes the *standard Neumann-Kirchhoff condition*

$$\partial_{\nu}^{\overline{p}} u(\mathrm{v}) = 0 \quad \text{for all } \mathrm{v} \in \mathrm{V(G)},$$

by taking $\omega_{\mathrm{v}} = 0$ for all v.

The elliptic problem on a metric graph.

Definition

We say that $v \in L^1(\mathsf{G})$ is a weak solution of problem (P^f_ω) if there exists $u \in \mathcal{W}^{1,\overline{p}}(\mathsf{G})$, with $v = \overline{\gamma}(u)$ a.e in G, such that

$$\int_{\mathsf{G}} v\varphi \, dx + \sum_{\mathrm{e}\in \mathrm{E}(\mathsf{G})} \int_0^{\ell_{\mathrm{e}}} |[u]'_{\mathrm{e}}|^{p_{\mathrm{e}}-2}[u]'_{\mathrm{e}}[\varphi]'_{\mathrm{e}} \, dx = \int_{\mathsf{G}} f\varphi \, dx + \sum_{\mathrm{v}\in \mathrm{V}(\mathsf{G})} \omega_{\mathrm{v}}\varphi(\mathrm{v}) \quad \forall \varphi \in \mathcal{W}^{1,\overline{p}}(\mathsf{G}). \tag{12}$$

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Note that given $\mathrm{e} \in E(\mathrm{G})$, if we take in (12) as test function φ such that $[\varphi]_{\mathrm{e}} = \phi$, with $\phi \in \mathcal{D}(]0,\ell_{\mathrm{e}}[)$ and $[\varphi]_{\mathrm{e}'} = 0$ for all $\mathrm{e}' \neq \mathrm{e}$, then we obtain that

$$[v]_{\mathrm{e}} = (|[u]'_{\mathrm{e}}|^{p_{\mathrm{e}}-2}[u]'_{\mathrm{e}})' + [f]_{\mathrm{e}} \quad \text{in} \quad \mathcal{D}'(]0,\ell_{\mathrm{e}}[). \tag{13}$$

Moreover, by Green's formula, the Kirchhoff condition

$$\partial^{\overline{p}}_{\nu} u(\mathrm{v}) = \omega_{\mathrm{v}} \quad \forall \mathrm{v} \in V(\mathrm{G})$$

is satisfied point–wise.

The elliptic problem on a metric graph.

Proposition

If v *is weak solutions to problem* (P^f_ω), *then, for all* $\mathrm{e} \in E(\mathrm{G})$, *there exist* $a_{\mathrm{e}}, b_{\mathrm{e}} \in \mathbb{R}$, *such that* $[v_{\mathrm{e}}]$ *is weak solution to problem* $(P^{\gamma_{\mathrm{e}},[f]_{\mathrm{e}}}_{p_{\mathrm{e}},a_{\mathrm{e}},b_{\mathrm{e}}})$.
Moreover, for any $\mathrm{v} \in V(\mathrm{G})$, *we have*

$$\sum_{\mathrm{e}:\mathrm{i}_{\mathrm{e}}=\mathrm{v}} a_{\mathrm{e}} + \sum_{\mathrm{e}:\mathrm{f}_{\mathrm{e}}=\mathrm{v}} b_{\mathrm{e}} = \omega_{\mathrm{v}}. \tag{14}$$

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$$\sum_{\mathrm{e}:\mathrm{i}_\mathrm{e}=\mathrm{v}} a_\mathrm{e} + \sum_{\mathrm{e}:\mathrm{f}_\mathrm{e}=\mathrm{v}} b_\mathrm{e} = \omega_\mathrm{v}. \tag{14}$$

By defining the following $\overline{p}$-Laplacian operator $\Delta^{\mathrm{G},\omega}_{\overline{p}}$ on functions $u \in \mathcal{W}^{1,\overline{p}}(\mathrm{G})$ as:

$$\int_\mathrm{G} -\Delta^{\mathrm{G},\omega}_{\overline{p}} u(x)\varphi(x) = \sum_{\mathrm{e}\in \mathrm{E}(\mathrm{G})} \int_0^{\ell_\mathrm{e}} |[u]'_\mathrm{e}|^{p_\mathrm{e}-2}[u]'_\mathrm{e}[\varphi]'_\mathrm{e}\, dx - \sum_{\mathrm{v}\in \mathrm{V}(\mathrm{G})} \omega_\mathrm{v}\varphi(\mathrm{v}) \quad \forall\, \varphi \in \mathcal{W}^{1,\overline{p}}(\mathrm{G}),$$

Problem (10) can be written as

$$(P^f_\omega) \quad \begin{cases} v - \Delta^{\mathrm{G},\omega}_{\overline{p}} u = f & \text{in } \mathrm{G}, \\ v = \overline{\gamma}(u) & \text{in } \mathrm{G}, \\ u \text{ continuous}, \end{cases}$$

The elliptic problem on a metric graph.

Theorem

Let G *be a finite and connected metric graph. Let* $1 < p_{\mathsf{e}} < \infty$ *be and* $\gamma_{\mathsf{e}} : \mathbb{R} \to \mathbb{R}$ *be a continuous and increasing function with* $\gamma_{\mathsf{e}}(\mathbb{R}) = \mathbb{R}$ *and* $\gamma_{\mathsf{e}}(0) = 0$ *for all* $\mathsf{e} \in E(\mathsf{G})$. *For* v_i *weak solutions to* $(P^{f_i}_{\omega^i})$, $i = 1, 2$, *we have*

$$\int_{\mathsf{G}} (v_1 - v_2)^+ \leq \int_{\mathsf{G}} (f_1 - f_2)^+ + \sum_{\mathsf{v} \in V(\mathsf{G})} (\omega^1_{\mathsf{v}} - \omega^2_{\mathsf{v}})^+ .$$

The elliptic problem on a metric graph.

Lemma

Assume $\gamma : \mathbb{R} \to \mathbb{R}$ *is a continuous and increasing function with* $\gamma(\mathbb{R}) = \mathbb{R}$ *and* $\gamma(0) = 0$. *We have the statements:*

(i) *Let* $v_\epsilon = \gamma(u_\epsilon)$ *be weak solution to problem* $(P^{\gamma,f}_{p,a,b+\epsilon})$, *with* $\epsilon \in \mathbb{R}$. *Then:*
1. For $\epsilon > 0$, *we have have* $u_0 \leq u_\epsilon$ *and* $u_0(\ell) < u_\epsilon(\ell)$.
And for $\epsilon < 0$, *we have that* $u_\epsilon \leq u_0$, *and* $u_\epsilon(\ell) < u_0(\ell)$.
2. Moreover, $\epsilon \mapsto u_\epsilon(x)$ *is continuous in* $(-\infty, +\infty)$ *for all fixed* $x \in [0, \ell]$, *and*

$$\lim_{\epsilon \to +\infty} u_\epsilon(\ell) = +\infty,$$

$$\lim_{\epsilon \to -\infty} u_\epsilon(\ell) = -\infty.$$

The elliptic problem on a metric graph.

Lemma

(ii) *Let* $v_\epsilon = \gamma(u_\epsilon)$ *be weak solution to problem* $(P^{\gamma,f}_{p,a+\epsilon,b})$*, with* $\epsilon \in \mathbb{R}$*. Then:*
1. For $\epsilon > 0$*, we have have that* $u_0 \leq u_\epsilon$ *and* $u_0(0) < u_\epsilon(0)$
And for $\epsilon < 0$*, we have that* $u_\epsilon \leq u_0$*.*
2. Moreover, $\epsilon \mapsto u_\epsilon(x)$ *is continuous in* $(-\infty, +\infty)$ *for all fixed* $x \in [0,\ell]$*, and*

$$\lim_{\epsilon\to+\infty} u_\epsilon(0) = +\infty, \quad \textit{and} \quad \lim_{\epsilon\to-\infty} u_\epsilon(0) = -\infty.$$

(iii) *Let* $v_\epsilon = \gamma(u_\epsilon)$ *be weak solution of* $(P^{\gamma,f}_{p,a-\epsilon,b+\epsilon})$ *for* $\epsilon \geq 0$*.*
1. For $\epsilon > 0$ *we have* $u_\epsilon(\ell) > u_0(\ell)$ *and* $u_\epsilon(0) < u_0(0)$*. For* $\epsilon < 0$ *we have* $u_\epsilon(\ell) < u_0(\ell)$*, and* $u_\epsilon(0) > u_0(0)$*. 2. Moreover,* $\epsilon \mapsto u_\epsilon(\ell)$ *and* $\epsilon \mapsto u_\epsilon(0)$ *are continuous in* $(-\infty, +\infty)$*.*

The elliptic problem on a metric graph.

Lemma

Let G *be a connected and compact metric graph with* $\mathsf{V}(\mathsf{G}) = \{\mathsf{v}_1, \ldots, \mathsf{v}_{n-1}, \mathsf{v}_n\}$*. Assume* $\gamma_{\mathsf{e}} : \mathbb{R} \to \mathbb{R}$ *is a continuous and increasing function with* $\gamma_{\mathsf{e}}(\mathbb{R}) = \mathbb{R}$ *and* $\gamma_{\mathsf{e}}(0) = 0$ *for all* $\mathsf{e} \in \mathsf{E}(\mathsf{G})$*, and, for* $\{\omega_{\mathsf{v}_1}, \ldots, \omega_{\mathsf{v}_{n-1}}, \omega_{\mathsf{v}_n}\} \subset \mathbb{R}$ *and* $\epsilon \in \mathbb{R}$*, set* ω^ϵ *be such that*

$$\omega^\epsilon_{\mathsf{v}_i} = \omega_{\mathsf{v}_i}, \ \ i = 1, 2, ..., n-1,$$

$$\omega^\epsilon_{\mathsf{v}_n} = \omega_{\mathsf{v}_n} + \epsilon.$$

Let $v_\epsilon = \overline{\gamma}(u_\epsilon)$ *be weak solution to problem* $(P^f_{\omega_\epsilon})$ *Then, we have that* $\epsilon \mapsto u_\epsilon(\mathsf{v}_n)$ *is continuous, and*

$$u_0 \leq u_\epsilon \quad \textit{if} \quad \epsilon \geq 0, \quad \textit{and} \quad u_\epsilon \leq u_0 \quad \textit{if} \quad \epsilon \leq 0.$$

Moreover

$$\epsilon \mapsto u_\epsilon(\mathsf{v}_n) \ \ \textit{is continuous.}$$

The elliptic problem on a metric graph.

Lemma

Let G *be a connected and compact metric graph with*

$$V(\mathsf{G}) = \{\mathsf{v}_1, \mathsf{v}_2 \ldots, \mathsf{v}_{n-1}, \mathsf{v}_n\}$$

Assume $\gamma_{\mathsf{e}} : \mathbb{R} \to \mathbb{R}$ *is a continuous and increasing function with* $\gamma_{\mathsf{e}}(\mathbb{R}) = \mathbb{R}$ *and* $\gamma_{\mathsf{e}}(0) = 0$ *for all* $\mathsf{e} \in E(\mathsf{G})$*, and assume*

$$\mathsf{v}_1, \mathsf{v}_n \in \partial \mathsf{V}(\mathsf{G}),$$

$$\mathsf{v}_1 = \mathsf{i}_{\mathsf{e}_1}, \; \mathsf{v}_n = \mathsf{f}_{\mathsf{e}_2}.$$

For $\{\omega_{\mathsf{v}_1}, \omega_{\mathsf{v}_2}, \ldots, \omega_{\mathsf{v}_{n-1}}, \omega_{\mathsf{v}_n}\} \subset \mathbb{R}$ *and* $\epsilon \in \mathbb{R}$*, set* ω^ϵ *be such that*

$$\omega^\epsilon_{\mathsf{v}_1} = \omega_{\mathsf{v}_1} - \epsilon,$$

$$\omega^\epsilon_{\mathsf{v}_i} = \omega_{\mathsf{v}_i}, \; i = 2, 3, ..., n-1,$$

$$\omega^\epsilon_{\mathsf{v}_n} = \omega_{\mathsf{v}_n} + \epsilon.$$

Then, we have

The elliptic problem on a metric graph.

Lemma

(i) *Let* $v_\epsilon = \overline{\gamma}(u_\epsilon)$ *be weak solution to problem* $(P^f_{\omega_\epsilon})$, *with* $\epsilon \geq 0$. *Then,* $\epsilon \to [u_\epsilon]_{\mathrm{e}_2}(\ell_{\mathrm{e}_2})$ *is continuous and increasing in* $[0, +\infty)$, $\epsilon \to [u_\epsilon]_{\mathrm{e}_1}(0)$ *is continuous and decreasing in* $[0, +\infty)$, *and*

$$\lim_{\epsilon \to +\infty} [u_\epsilon]_{\mathrm{e}_2}(\ell_{\mathrm{e}_2}) = +\infty \quad \textit{or} \quad \lim_{\epsilon \to +\infty} [u_\epsilon]_{\mathrm{e}_1}(0) = -\infty. \tag{15}$$

(ii) *Let* $v_\epsilon = \overline{\gamma}(u_\epsilon)$ *be weak solution to problem* $(P^f_{\omega_\epsilon})$, *with* $\epsilon \leq 0$. *Then,* $\epsilon \to [u_\epsilon]_{\mathrm{e}_2}(\ell_{\mathrm{e}_2})$ *is continuous and decreasing in* $(-\infty, 0])$, $\epsilon \to [u_\epsilon]_{\mathrm{e}_1}(0)$ *is continuous and increasing in* $(-\infty, 0]$, *and*

$$\lim_{\epsilon \to -\infty} [u_\epsilon]_{\mathrm{e}_2}(\ell_{\mathrm{e}_2}) = -\infty \quad \textit{or} \quad \lim_{\epsilon \to -\infty} [u_\epsilon]_{\mathrm{e}_1}(0) = +\infty. \tag{16}$$

The elliptic problem on a metric graph.

Theorem

Let G *be a finite and connected metric graph. Let* $1 < p_{\mathsf{e}} < \infty$ *be and* $\gamma_{\mathsf{e}} : \mathbb{R} \to \mathbb{R}$ *be a continuous and increasing function with* $\gamma_{\mathsf{e}}(\mathbb{R}) = \mathbb{R}$ *and* $\gamma_{\mathsf{e}}(0) = 0$ *for all* $\mathsf{e} \in E(\mathsf{G})$*. Then, for all* $f \in L^1(\mathsf{G})$ *and all* $\omega = \{\omega_{\mathsf{v}} : \mathsf{v} \in \mathrm{V}(\mathsf{G})\} \subset \mathbb{R}$*, there exists a unique weak solution to the problem* (P_ω^f)*.*

The elliptic problem on a metric graph.

Theorem

Let G *be a finite and connected metric graph. Let* $1 < p_{\mathrm{e}} < \infty$ *be and* $\gamma_{\mathrm{e}} : \mathbb{R} \to \mathbb{R}$ *be a continuous and increasing function with* $\gamma_{\mathrm{e}}(\mathbb{R}) = \mathbb{R}$ *and* $\gamma_{\mathrm{e}}(0) = 0$ *for all* $\mathrm{e} \in E(\mathrm{G})$*. Then, for all* $f \in L^1(\mathrm{G})$ *and all* $\omega = \{\omega_{\mathrm{v}} : \mathrm{v} \in \mathrm{V}(\mathrm{G})\} \subset \mathbb{R}$*, there exists a unique weak solution to the problem* (P_{ω}^{f})*.*

Sketch of proof

The existence is by induction on the number N of edges

The elliptic problem on a metric graph.

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The elliptic problem on a metric graph.

Theorem

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Sketch of proof

The existence is by induction on the number N of edges

We know is true for $N = 1$.

$$\mathrm{E}(\mathrm{G}_{N+1}) = \{\mathrm{e}_1, \ldots, \mathrm{e}_N, \mathrm{e}_{N+1}\} \quad V(\mathrm{G}_{N+1}) = \{\mathrm{v}_1, \ldots, \mathrm{v}_m\}$$

The elliptic problem on a metric graph.

Theorem

Let G *be a finite and connected metric graph. Let* $1 < p_{\mathrm{e}} < \infty$ *be and* $\gamma_{\mathrm{e}} : \mathbb{R} \to \mathbb{R}$ *be a continuous and increasing function with* $\gamma_{\mathrm{e}}(\mathbb{R}) = \mathbb{R}$ *and* $\gamma_{\mathrm{e}}(0) = 0$ *for all* $\mathrm{e} \in E(\mathrm{G})$. *Then, for all* $f \in L^1(\mathrm{G})$ *and all* $\omega = \{\omega_{\mathrm{v}} : \mathrm{v} \in \mathrm{V}(\mathrm{G})\} \subset \mathbb{R}$, *there exists a unique weak solution to the problem* (P^f_ω).

Sketch of proof

The existence is by induction on the number N of edges

We know is true for $N = 1$.

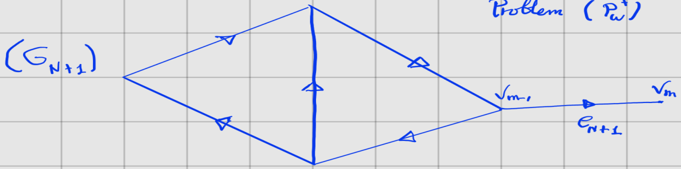
$$\mathrm{E}(\mathrm{G}_{N+1}) = \{\mathrm{e}_1, \ldots, \mathrm{e}_N, \mathrm{e}_{N+1}\} \quad V(\mathrm{G}_{N+1}) = \{\mathrm{v}_1, \ldots, \mathrm{v}_m\}$$

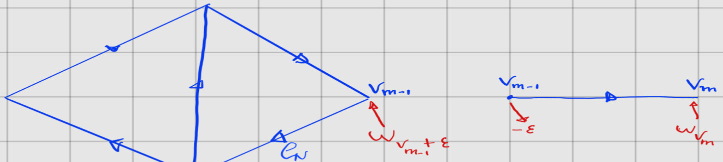
(a) The graph G_{N+1} has a vertex of degree 1

(b) All the vertices of G_{N+1} have degree larges or equal to 2

The elliptic problem on a metric graph.

Problem (P_ω^f)



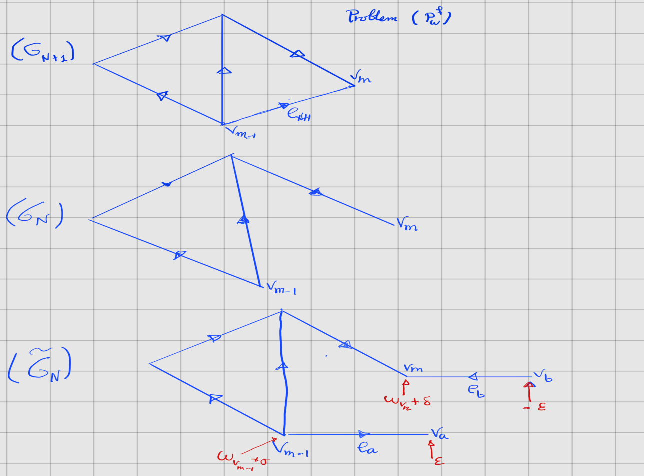


Lemmas $\longrightarrow$ $\exists\ \varepsilon > 0$: $[\tilde{u}_\varepsilon]_{e_N}(v_{m-1}) = u_\varepsilon(v_{m-1})$

Gluing these solutions $\rightsquigarrow$ the unique solution of (P_ω^f)

The elliptic problem on a metric graph.

Problem (P_ω^f)



$P_{e_a} = P_{N+1}$ $\gamma_{e_a} = \gamma_{N+1}$ $[f]_{e_a} = [f]_{e_{N+1}}$

$P_{e_b} = P_{N+1}$ $\gamma_b = \gamma_{N+1}$

$[f]_{e_b}(x) = [f]_{e_{N+1}}\left(x + \frac{\ell_{e_{N+1}}}{2}\right)$

$\ell_{e_a} = \frac{\ell_{e_{N+1}}}{2}$, $\ell_{e_b} = \frac{\ell_{e_{N+1}}}{2}$

The parabolic problem on a metric graph.

We can rewrite Problem (5) as

$$(5)\qquad \begin{cases} \dfrac{\partial v}{\partial t} - \Delta_{\overline{p}}^{\mathsf{G},\omega} u = f \quad \text{in } (0,+\infty)\times \mathsf{G}, \\ v = \overline{\gamma}(u) \quad \text{in } (0,+\infty)\times \mathsf{G}, \\ u \text{ continuous}, \\ v(0) = v_0. \end{cases}$$

The parabolic problem on a metric graph.

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Set $X := L^1(\mathrm{G}) \times L^1(\mathrm{V}(\mathrm{G}))$. Let $f \in L^1(0,T;L^1(\mathrm{G}))$ and $\omega \in L^1(0,T;L^1(\mathrm{V}(\mathrm{G})))$. Problem (5), posed in $(0,T)$, can be rewritten as the following abstract Cauchy problem in X:

$$\begin{cases} W'(t) + \mathcal{A}(W(t)) \ni (f(t),\omega(t)) & t \in (0,T), \\ W(0) = (v_0, 0) & v_0 \in L^1(\mathrm{G}), \end{cases} \tag{17}$$

being $\mathcal{A}$ the operator:

$$\mathcal{A} = \{((v,0),(\widetilde{v},\omega)) \in X \times X : v \text{ is a solution of } (P_\omega^{v+\widetilde{v}})\}.$$

The parabolic problem on a metric graph.

We have that

$$((v,0),(\widetilde{v},\omega)) \in \mathcal{A} \iff \text{there exist } u \in \mathcal{W}^{1,\overline{p}}(\mathrm{G}),\ v = \gamma(u) \ \text{ a.e. in } \mathrm{G},$$

such that

$$-\Delta_{\overline{p}}^{\mathrm{G},\omega} u = \tilde{v}\,,$$

that is

$$\sum_{\mathrm{e}\in \mathrm{E}(\mathrm{G})} \int_0^{\ell_\mathrm{e}} |[u]'_\mathrm{e}|^{p_\mathrm{e}-2}[u]'_\mathrm{e}[\varphi]'_\mathrm{e}\, dx = \int_\mathrm{G} \widetilde{v}\varphi\, dx + \sum_{\mathrm{v}\in \mathrm{V}(\mathrm{G})} \omega_\mathrm{v}\varphi(\mathrm{v}) \quad \forall\, \varphi \in \mathcal{W}^{1,\overline{p}}(\mathrm{G}).$$

The parabolic problem on a metric graph.

Theorem

Under the assumptions of Theorem 12 we have that the operator $\mathcal{A}$ is m-T-accretive and satisfies

$$\overline{D(\mathcal{A})} = L^1(\mathsf{G}) \times \{0\}.$$

The parabolic problem on a metric graph.

Theorem

Under the assumptions of Theorem 12 we have that the operator $\mathcal{A}$ *is m-T-accretive and satisfies*

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Theorem

Let G *be a finite and connected metric graph. Let* $1 < p_{\mathrm{e}} < \infty$ *be and* $\gamma_{\mathrm{e}} : \mathbb{R} \to \mathbb{R}$ *be a continuous and increasing function with* $\gamma_{\mathrm{e}}(\mathbb{R}) = \mathbb{R}$ *and* $\gamma_{\mathrm{e}}(0) = 0$ *for all* $\mathrm{e} \in E(\mathrm{G})$. *Let* $f \in L^1_{loc}([0, +\infty[; L^1(\mathrm{G}))$ *and* $\omega \in L^1_{loc}([0, +\infty[; L^1(\mathrm{V}(\mathrm{G})))$ *be. Then, for every initial data* $v_0 \in L^1(\mathrm{G})$, *there exists a unique mild solution of Problem* (17) *on* $[0, T]$, *for any* $T > 0$.

The parabolic problem on a metric graph.

Definition

Let $f \in L^1_{loc}([0,+\infty[; L^1(\mathrm{G}))$ and $\omega \in L^1_{loc}([0,+\infty[; L^1(\mathrm{V}(\mathrm{G})))$ be. We say that $v : [0,+\infty[\to L^1(\mathrm{G})$ is a weak solution of Problem (5) if, for any $T > 0$, $v \in C([0,T]; L^1(\mathrm{G}))$, $v(0) = v_0$ and there exists $u \in L^{\overline{p}}(0,T; \mathcal{W}^{1,\overline{p}}(\mathrm{G}))$ with $v(t) = \overline{\gamma}(u(t))$ for all $t \in [0,T]$ satisfying

$$-\int_0^T \int_{\mathrm{G}} v(t) \frac{\partial}{\partial t}\psi(t)\, dx\, dt + \sum_{\mathrm{e}\in \mathrm{E}(\mathrm{G})} \int_0^T \int_0^{\ell_{\mathrm{e}}} |[u(t)]'_{\mathrm{e}}|^{p_{\mathrm{e}}-2}[u(t)]'_{\mathrm{e}}[\psi(t)]'_{\mathrm{e}}\, dx\, dt$$
$$= \int_0^T \int_{\mathrm{G}} f(t)\psi(t)\, dx\, dt + \sum_{\mathrm{v}\in \mathrm{V}(\mathrm{G})} \int_0^T \omega_{\mathrm{v}}(t)\psi(t,\mathrm{v})\, dt,$$

for all $\psi \in W^{1,1}(0,T; \mathcal{W}^{1,1}(\mathrm{G})) \cap L^{\overline{p}}(0,T; \mathcal{W}^{1,\overline{p}}(\mathrm{G}))$ with $\psi(0) = \psi(T) = 0$.

The parabolic problem on a metric graph.

Theorem

Let G *be a finite and connected metric graph. Let* $1 < p_{\mathsf{e}} < \infty$ *be and* $\gamma_{\mathsf{e}} : \mathbb{R} \to \mathbb{R}$ *be a continuous and increasing function with* $\gamma_{\mathsf{e}}(\mathbb{R}) = \mathbb{R}$ *and* $\gamma_{\mathsf{e}}(0) = 0$ *for all* $\mathsf{e} \in E(\mathsf{G})$*. Let* $v_0 \in L^1(\mathsf{G})$*,* $f \in L^1_{loc}([0, +\infty[; L^1(\mathsf{G}))$ *and* $\omega \in L^1_{loc}([0, +\infty[; L^1(\mathsf{V}(\mathsf{G})))$ *be. If* v *is a weak solution of problem* (5)*, then* $W := (v, 0)$ *is an mild solution of problem* (17)*.*

The parabolic problem on a metric graph.

Theorem

Let G *be a finite and connected metric graph. Let* $1 < p_{\mathrm{e}} < \infty$ *be and* $\gamma_{\mathrm{e}} : \mathbb{R} \to \mathbb{R}$ *be a continuous and increasing function with* $\gamma_{\mathrm{e}}(\mathbb{R}) = \mathbb{R}$ *and* $\gamma_{\mathrm{e}}(0) = 0$ *for all* $\mathrm{e} \in E(\mathrm{G})$*. Let* $v_0 \in L^1(\mathrm{G})$*,* $f \in L^1_{loc}([0,+\infty[; L^1(\mathrm{G}))$ *and* $\omega \in L^1_{loc}([0,+\infty[; L^1(\mathrm{V}(\mathrm{G})))$ *be. If* v *is a weak solution of problem* (5)*, then* $W := (v, 0)$ *is an mild solution of problem* (17)*.*

For $\beta : \mathbb{R} \to \mathbb{R}$ increasing,

$$j_\beta(r) := \int_0^r \beta(s)\, ds$$

defines a convex lower semi-continuous function such that $\beta = \partial j_\gamma$, the subdifferential of j_β. Let j^*_β be the Legengre transform of j_β, then

$$\beta^{-1} = \partial j^*_\beta.$$

In the next result we use the following notation: for $u \in L^1(\mathrm{G})$, $j^*_{\vec{\gamma}}(u)$ is defined as

$$[j^*_{\vec{\gamma}}(u)]_{\mathrm{e}} = j^*_{\gamma_{\mathrm{e}}}([u]_{\mathrm{e}}) \quad \text{for all } \mathrm{e} \in E(\mathrm{G}).$$

The parabolic problem on a metric graph.

Theorem

Let G *be a finite and connected metric graph. For each* $\mathrm{e} \in \mathrm{E}(\mathrm{G})$, *let* $1 < p_{\mathrm{e}} < \infty$ *be and* $\gamma_{\mathrm{e}} : \mathbb{R} \to \mathbb{R}$ *be a continuous and increasing function with* $\gamma_{\mathrm{e}}(\mathbb{R}) = \mathbb{R}$ *and* $\gamma_{\mathrm{e}}(0) = 0$. *Let* $f \in L^{\overline{p'}}_{loc}([0, +\infty[; L^{\overline{p'}}(\mathrm{G}))$ *and* $\omega \in L^{\infty}_{loc}([0, +\infty[; L^1(\mathrm{V}(\mathrm{G})))$ *be. For every initial data* $v_0 \in L^1(\mathrm{G})$ *with* $\displaystyle\int_{\mathrm{G}} j^*_{\overline{\gamma}}(v_0) < +\infty$, *there exists a unique weak solution of problem* (5).

Moreover, if, for $i = 1, 2$, $f_i \in L^{\overline{p'}}_{loc}([0, +\infty[; L^{\overline{p'}}(\mathrm{G}))$, $\omega_i \in L^{\overline{p'}}_{loc}([0, +\infty[; L^1(\mathrm{V}(\mathrm{G})))$, *and* v_i *is weak solution for initial data* $v_{i,0} \in L^1(\mathrm{G})$, *with* $\displaystyle\int_{\mathrm{G}} j^*_{\overline{\gamma}}(v_{i,0}) < +\infty$, *then, for any* $t > 0$:

$$\int_{\mathrm{G}} (v_1(t) - v_2(t))^+ \leq \int_{\mathrm{G}} (v_{1,0} - v_{2,0})^+ + \int_0^t \int_{\mathrm{G}} (f_1(s) - f_2(s))^+ \, dxds$$

$$+ \sum_{\mathrm{v} \in \mathrm{V}(\mathrm{G})} \int_0^t (\omega_1(s, \mathrm{v}) - \omega_2(s, \mathrm{v}))^+ \, ds.$$