Serrin's overdetermined problems for Hessian operators: symmetry and stability

Gloria Paoli

Università degli Studi di Napoli Federico II

Joint work with N. Gavitone, A.L. Masiello, and G. Poggesi

Multiscale Stochastics, Patterns, and Analysis of Combinatorial Environments

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The Soap Bubble Theorem

SBT [A. D. Aleksandrov, *V. Amer. Math. Soc.* 1958]

A C^2-smooth compact hypersurface S embedded in $\mathbb{R}^n$ with constant mean curvature H is a sphere.

- Proof by the **Method of Moving Plane (MMP)** or Reflection Principle: Use of the strong maximum principle and the Hopf boundary point lemma.
- Recall: H at $p \in S$ is defined as the arithmetic mean of its principal curvatures at p.

Serrin's overdetermined Poisson problem

Theorem [J. Serrin, *Arch. Rational Mech. Anal.*, 1971]

Let Ω be a bounded, smooth, connected, open set in $\mathbb{R}^n$ and $u \in C^2(\overline{\Omega})$ a solution to

$$\begin{cases} \Delta u = n & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ u_\nu = R & \text{on } \partial\Omega, \end{cases} \tag{OP}$$

where ν is the unit outer normal, and $R > 0$. Then, up to translation, Ω is a ball of radius R and $u(x) = \frac{|x|^2 - R^2}{2}$.

Physical motivation: related to the torsion of a solid bar and the tangential stress of a fluid on the walls of a rectilinear pipe.

Serrin's overdetermined problem: some bibliography

- [**Serrin, *Arch. Ration. Mech. Anal.*, 1971**] Use of a version of MMP and a refinement of the maximum principle. Adaptable to a range of non-linear elliptic operators with general data. It requires C^2 regularity up to the boundary.

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- [**Weinberger, *Arch. Ration. Mech. Anal.*, 1971**] Use of the maximum principle applied to an auxiliary function and the integral Pohožaev's identity.
  - It requires less regularity: $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$.
  - Payoff: Relies on the linearity of the operator and the constant r-h-s of the Poisson problem.

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- [**Brandolini-Nitsch-Salani-Trombetti, *Arch. Ration. Mech. Anal.*, 2008**] Alternative proof: integral approach via the arithmetic-geometric mean inequality and Pohožaev identity.
    - Regularity required: $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$.
    - Extension of the rigidity result to overdetermined problems for Hessian operators.

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  - Extension of the rigidity result to overdetermined problems for Hessian operators.

Weinberger's P function method

Let $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ solution to

$$\begin{cases} \Delta u = n & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

We define $P = \frac{|\nabla u|^2}{2} - u$ and we have

$$\Delta P = \sum_{i,j}^{n} \left(\frac{\partial^2 u}{\partial x_i \partial x_j} \right)^2 - n$$

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$$\Delta P = \sum_{i,j}^{n} \left(\frac{\partial^2 u}{\partial x_i \partial x_j} \right)^2 - n \geq 0$$

by Cauchy-Schwarz:

$$\sum_{i,j}^{n} \left(\frac{\partial^2 u}{\partial x_i \partial x_j} \right)^2 \geq \sum_{i}^{n} \left(\frac{\partial^2 u}{\partial x_i^2} \right)^2 \geq \frac{1}{n} \left(\sum_{i}^{n} \frac{\partial^2 u}{\partial x_i^2} \right)^2 = \frac{(\Delta u)^2}{n} = n$$

Weinberger's P function method

$$P = \frac{|\nabla u|^2}{2} - u, \qquad \Delta P = \left|D^2 u\right|^2 - \frac{(\Delta u)^2}{n} \geq 0$$

The idea is to use ΔP as a "spherical detector"

$$\Delta P = 0 \quad \Longleftrightarrow \quad \Omega \text{ is a ball.}$$

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How to determine a? From the condition $u = 0$ on $\partial\Omega$, we have $a = R^2$. Consequently, Ω is a ball centered at x_0 of radius R.

Weinberger's proof of the overdetermined problem

How to use the P-function to prove

Theorem [J. Serrin, *Arch. Rational Mech. Anal.*, 1971]

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where ν is the unit outer normal, and $R > 0$. Then, up to translation, Ω is a ball of radius R and $u(x) = \frac{|x|^2 - R^2}{2}$.

How to prove that $u_\nu = R$ implies $\Delta P = 0$?

Weinberger's proof of the overdetermined problem

- By the maximum principle and the overdetermined condition $|\nabla u| = R$ on $\partial\Omega$

$$\max_{\overline{\Omega}} P = \max_{\partial\Omega} P = \max_{\partial\Omega} \left(\frac{|\nabla u|^2}{2} - u \right) = \frac{R^2}{2}.$$

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- Recall Pohažaev identity

$$\int_{\Omega} |\nabla u|^2 dx = \int_{\partial\Omega} u_\nu^2 (x, \nu) d\sigma$$

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- We have

$$\int_\Omega \left(\frac{R^2}{2} - P \right) dx = 0.$$

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$$\int_\Omega \left(\frac{R^2}{2} - P \right) dx = 0.$$

- $P \equiv \frac{R^2}{2}$ on Ω, $\Rightarrow \Delta P = 0$.

Aleksandrov’s SBT: some bibliography

- [**A. D. Aleksandrov, *V. Amer. Math. Soc.* 1958**] Use of the MMP and the maximum principle. It requires a C^2 embedded surface.

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- **[A. D. Aleksandrov, *V. Amer. Math. Soc.* 1958]** Use of the MMP and the maximum principle. It requires a C^2 embedded surface.
- **[R. C. Reilly, *Indiana Univ. Math. J*, 1977]** Use of the maximum principle applied to an auxiliary function and integral identities.
  - It make use of the P-function method by Weinberger.

After Serrin & Weinberger papers...

- Garofalo & Lewis 1989;
- Henrot & Philippin 1998.
- Brock & Henrot 2002;
- Fragalà, Gazzola & Kawohl 2006;
- Brandolini, Nitsch, Salani & Trombetti 2008.
- Cianchi & Salani 2011.
- Brandolini, Gavitone, Nitsch & Trombetti 2014.
- Gavitone & Molinarolo 2026
- Ciraolo, Magnanini & Sakaguchi 2016
- Nitsch & Trombetti 2018
- Tralli & Pacella 2019
- Magnanini & Poggesi, 2019
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SBT: some references...

- Liebmann 1899
- Suss 1952
- Hsiung 1954
- Alexandrov 1958;
- Reilly, 1977
- Ros, 1987
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- Ciraolo & Maggi 2017
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There is a connection between the two problems!

The connection between Alexandrov and Serrin theorems

Magnanini-Poggesi, 2019

Let Ω be a bounded domain of class C^2, let u be the torsion function of Ω. and let

$$R = \frac{n|\Omega|}{P(\Omega)}, \qquad H_0 = \frac{1}{R} = \frac{P(\Omega)}{n|\Omega|}$$

Then

$$\frac{1}{n-1}\int_\Omega \Delta P\,dx + \frac{1}{R}\int_{\partial\Omega}(u_\nu - R)^2 = \int_{\partial\Omega}(H_0 - H)u_\nu^2$$

$\boxed{\text{Rigidity in both problems:}}$ $H \equiv H_0$ implies $u_\nu \equiv R$

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Rigidity in both problems: $H \equiv H_0$ implies $u_\nu \equiv R$

Stability in both problems: $\|u_\nu - R\|_2 \leq C\|H_0 - H\|_2.$

The Hessian Operator

S_k-operator

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$a_{ij} = a_{ji}, \quad i, j = 1, 2, 3$$

$$S_1(A) = \mathrm{tr}(A) = (a_{11} + a_{22} + a_{33})$$

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S_k-operators

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

$$a_{ij} = a_{ji}, \quad \textit{i,j=1,2,3}$$

- ◇ $S_1(A) = \mathrm{tr}(A) = (a_{11} + a_{22} + a_{33}) = \lambda_1 + \lambda_2 + \lambda_3$
- ◇ $S_2(A) = (a_{11}a_{22} - a_{12}a_{21}) + (a_{12}a_{23} - a_{13}a_{22}) + (a_{21}a_{32} - a_{22}a_{31}) + (a_{22}a_{33} - a_{23}a_{32}) + (a_{11}a_{33} - a_{13}a_{31}) + (a_{11}a_{32} - a_{12}a_{31}) + (a_{11}a_{23} - a_{13}a_{21}) + (a_{21}a_{33} - a_{23}a_{31}) + (a_{12}a_{33} - a_{13}a_{32})$
  $= \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3$
- ◇ $S_3(A) = \det(A) = \lambda_1\lambda_2\lambda_3$

S_k-operators, $k = 1, \dots, n$

Let $(\lambda_1, \dots, \lambda_n)$ be the eigenvalues of a symmetric matrix $A \in \mathbb{S}^{n\times n}$, then

$\boxed{k\text{−th elementary symmetric function}}$

$$S_k(A) = S_k(\lambda_1, \dots, \lambda_n) = \sum_{1\le i_1<\cdots<i_k\le n} \lambda_{i_1} \dots \lambda_{i_k}.$$

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Let $\Omega \subset \mathbb{R}^n$, $u \in C^2(\Omega)$ and D^2u the Hessian matrix.

$k-$th Hessian operator $\quad S_k(D^2u)$

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- $S_1(D^2u) = \Delta u$
- $S_n(D^2u) = \det(D^2u)$ Monge-Ampére operator

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- $S_1(D^2u) = \Delta u$
- $S_n(D^2u) = \det(D^2u)$ Monge-Ampére operator
- $S_k(D^2u)$ is a second-order fully-nonlinear differential operator.

S_k-operators, $k = 1, \dots, n$

Let $(\lambda_1, \dots, \lambda_n)$ be the eigenvalues of a symmetric matrix $A \in \mathbb{S}^{n\times n}$, then

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$k-$th Hessian operator $\quad S_k(D^2u)$

- $S_1(D^2u) = \Delta u$
- $S_n(D^2u) = \det(D^2u)$ Monge-Ampére operator
- Admissible functions (k- convex functions)

$$\Phi_k(\Omega) = \left\{ v \in C^2(\Omega) : S_i(D^2v) \ge 0 \ \text{ in } \ \Omega, \, i = 1, \dots, k \right\}$$

Some properties of S_k

Let $\Omega \subset \mathbb{R}^n$ be a smooth domain and let $u \in C^2(\Omega)$. Then

- **Divergence form:** $$S_k(D^2u) = \frac{1}{k}\left(S_k^{ij}(D^2u)u_j\right)_i$$

  where $S_k^{ij}(D^2u) = \dfrac{\partial S_k(D^2u)}{\partial u_{ij}}$

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- Divergence form: $$S_k(D^2u) = \frac{1}{k}\left(S_k^{ij}(D^2u)u_j\right)_i$$
- Newton's inequalities: let $u \in \Phi_n(\Omega)$, then

$$\frac{S_1(D^2u)}{n} \geq \cdots \geq \left(\frac{S_k(D^2u)}{\binom{n}{k}}\right)^{\frac{1}{k}} \geq \left(\frac{S_{k+1}(D^2u)}{\binom{n}{k+1}}\right)^{\frac{1}{k+1}} \geq \cdots \geq (S_n(D^2u))^{\frac{1}{n}}.$$

If there is equality at any stage, D^2u is a multiple of the identity.

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- Divergence form: $$S_k(D^2u) = \frac{1}{k}\left(S_k^{ij}(D^2u)u_j\right)_i$$

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- Let $u \in \Phi_k(\Omega)$, then
$$\left(\frac{S_{i-1}(D^2u)}{\binom{n}{i-1}}\right)^{\frac{i}{i-1}} \geq \frac{S_i(D^2u)}{\binom{n}{i}}, \quad i = 2, \ldots, (k+1)$$

The Serrin's problem for Hessian operators

Brandolini-Nitsch-Salani-Trombetti, 2008

Let $\Omega \subset \mathbb{R}^n$ be a smooth domain and let $u \in C^2(\overline{\Omega})$ the solution to

$$\begin{cases} S_k(D^2u) = \binom{n}{k} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ |\nabla u| = R & \text{on } \partial\Omega, \end{cases} \tag{k-S}$$

then, up to a translation, Ω is a ball of radius R and $u(x) = \dfrac{|x|^2 - R^2}{2}$

The proof is based on integral identities and Newton's inequalities.

Aims:

◇ Re-obtain the result via P-function method

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Aims:

- ◇ Re-obtain the result via P-function method
- ◇ Stability in the spirit of Magnanini Poggesi

Key fact of the P-Function

The Laplacian

- The auxiliary function $P = \dfrac{|\nabla u|^2}{2} - u$;
- The differential operator Δ for which the maximum principle holds;
- The Cauchy-Schwarz inequality and its rigidity;

The S_k operator

- The P-function will be expressed in terms of the solution to (k-OP)
- **A possibly linear differential operator for which the maximum principle holds?**
- Newton inequalities and their rigidity!

Rigidity result via the $P-$function

We give an alternative proof via the $P-$ function method ([N. Gavitone, A.L. Masiello, G. P., G. Poggesi, *arxiv*, 2025])

- We define the linear differential operator

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Rigidity result via the P-function

- We define

$$P = \frac{|\nabla u|^2}{2} - u.$$

- We can write

$$L[P] = n\binom{n}{k}\left[\left(\frac{\Delta u}{n} - \frac{S_k(D^2u)}{\binom{n}{k}}\right) + \frac{n-k}{n}\left(\frac{S_k(D^2u)}{\binom{n}{k}} - \frac{S_{k+1}(D^2u)}{\binom{n}{k+1}}\right)\right].$$

  By Newton's inequalities,

$$L[P](x) \geq 0 \quad \forall x \in \Omega, \quad \forall k = 1, \ldots, n$$

- $L[P]$ is the spherical detector.
  By Newton's inequalities, $L[P] = 0$ if and only if D^2u is the identity matrix, $1 \leq k < n$.
- Moreover, by the regularity theory for Hessian operators, the maximum principle holds for L.

Rigidity result via the P-function

- $\max_{\overline{\Omega}} P = \max_{\partial\Omega} P = \max_{\partial\Omega} \left(\frac{|\nabla u|^2}{2} - u \right) = \frac{R^2}{2}.$
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- $P \equiv \frac{R^2}{2}$ or $P < \frac{R^2}{2}$ on Ω.
- Using Pohažaev identity for the k-Hessian equation we can prove

$$\int_\Omega \left(\frac{R^2}{2} - P \right) dx = 0.$$

- $P \equiv \frac{R^2}{2}$ on $\Omega \Rightarrow L[P] = 0.$

Stability results

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- One way to measure the closeness of $\partial\Omega$ from a sphere is to fit Ω into a spherical annulus:

  $$B_{\rho_i}(z) \subset \Omega \subset B_{\rho_e}(z),$$

  where

  $$\rho_e - \rho_i \leq \psi(\eta),$$

  $\psi : [0, +\infty) \to [0, +\infty)$ is a continuous function vanishing at 0 and
  $\eta = ||H - H_0||$ for the soap bubble theorem
  $\eta = ||u_\nu - R||$ for the overdetermined problem
  for some suitable norm $|| \cdot ||$.

Brief overview on stability results for the Serrin problem

- **[Aftalion-Busca-Reichel, *Adv. Diff. Equat.*, 1999]**

$$\psi(\eta) = C|\log(\eta)|^{-1/n} \quad \text{and} \quad \eta := ||u_\nu - R||_{C^1(\partial\Omega)}$$

  - Proof holds also for $\Delta u = f(u)$
  - Use of a quantitative version of the MMP

- **[Brandolini-Nitsch-Salani-Trombetti, *J. Diff. Equat.* 2008]**

$$\psi(\eta) = C\eta^{\frac{1}{4n+9}} \quad \text{and} \quad \eta = \max_{\partial\Omega} |u_\nu - c| < \varepsilon,$$

- **[Magnanini-Poggesi, 2019]**

$$\psi(\eta) = C\eta^{\frac{2}{n+2}}, \quad \eta = ||u_\nu - R||_{2,\partial\Omega}$$

- **Other references:** [Ciraolo-Magnanini-Vespri, 2016], [Feldman, 2017].

Brief overview on stability results for the SBT

- **[Ciraolo-Maggi *Comm. Pure Appl. Math.*, 2017]**

$$\psi(\eta) = C\eta^{\frac{1}{2n+2}}, \quad \eta = ||H - H_0||_{\infty,\partial\Omega}$$

  - Restriction to strictly mean convex surfaces.

- **[Ciraolo-Vezzoni, *J. Eur. Math. Soc*, 2018]**

$$\psi(\eta) = C\eta, \quad \eta = \max_{\partial\Omega} H - \min_{\partial\Omega} H.$$

  - Optimal result: attained by ellipsoids. Proved by using MMP.

- **[Magnanini-Poggesi, 2019]**

$$\psi(\eta) = C\eta^{\tau_n}, \quad \eta = ||H - H_0||_{2,\partial\Omega}$$

  - with $\tau_n = 1$ for $n = 2, 3$ and $\tau_n = \frac{2}{n+2}$ for $n \geq 4$.

Stability for the k-Hessian problem

Stability of the spherical configuration

Let Ω be a bounded, smooth, connected, open set in $\mathbb{R}^n$. What can we say about Ω if

$$\begin{cases} S_k(D^2u) = \binom{n}{k} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ |\nabla u| \approx R & \text{on } \partial\Omega, \end{cases}$$

Overview

- **Case** $k = 1$ **(Laplace operator)**: Rigidity result and stability result for Serrin's overdetermined problem examined in the first part of the talk.
- **Case** $k = n$ **(Monge-Ampère operator)**:
  - Rigidity in [Brandolini-Nitsch-Salani-Trombetti, *Arch. Ration. Mech. Anal.*, 2008]. See also [Brandolini-Gavitone-Nitsch-Trombetti, *J. Math. Pures Appl.*, 2014]
  - Stability in [Brandolini-Nitsch-Salani-Trombetti, *Ann. Mat. Pura Appl*, 2009]
- **Case** $1 < k < n$:
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Our aim

We want to address the stability issue for Serrin's overdetermined problem in the case of the Hessian operators for $1 < k < n$ (recovering in our analysis also the cases $k = 1$ and $k = n$).

Stability result

Theorem [N. Gavitone, A.L. Masiello, P., G. Poggesi, *arxiv*, 2025]

Let $n \geq 2$ and $1 \leq k \leq n$. Let $\Omega \subset \mathbb{R}^n$ be a C^2, bounded, connected, open set and let $u \in C^2(\overline{\Omega})$ be the solution to

$$\begin{cases} S_k(D^2u) = \binom{n}{k} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Let z be a global minimum point of u in $\overline{\Omega}$ and set

$$R := \frac{1}{P(\Omega)} \int_{\partial\Omega} |\nabla u| d\mathcal{H}^{n-1}, \qquad \delta := \||\nabla u| - R\|_{L^\infty(\partial\Omega)}.$$

Then, we have that

$$\rho_e - \rho_i \leq C \begin{cases} \delta^{1/2} & \text{if } n = 2, \\ \delta^{1/2} \log\left(\frac{1}{\delta^{1/2}}\right) & \text{if } n = 3, \\ \delta^{\frac{1}{n-1}} & \text{if } n \geq 4. \end{cases}$$

Some remarks

- We recall that

$$\rho_e = \max_{x \in \partial\Omega} |x - z| \quad \text{and} \quad \rho_i = \min_{x \in \partial\Omega} |x - z|.$$

- The constant C can be explicitly estimated only in terms of n, the radius r_i of the uniform interior sphere condition of $\partial\Omega$, and the diameter d_Ω of Ω.

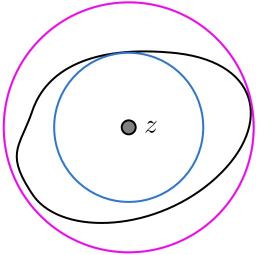

Figure: Definition of Inner and Outer radius

Key ingredient

- If Ω is a C^2 set, $(\kappa_1, \dots, \kappa_{n-1})$ the principal curvatures of $\partial\Omega$

$$H_k = \binom{n-1}{k}^{-1} \sum_{1 \le i_1 < \cdots < i_k \le n-1} \kappa_{i_1} \cdots \kappa_{i_k} = \frac{S_k(\kappa_1, \dots, \kappa_{n-1})}{\binom{n-1}{k}},$$

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  where $k = 1, \dots, n-1$

- We prove the following fundamental identity:

$$\binom{n}{k}^{-1} \frac{2}{k} \int_\Omega (-u) L[P] \, dx + n \int_\Omega (-u) \left[\frac{\Delta u}{n} - \frac{S_k(D^2 u)}{\binom{n}{k}} \right] dx = \frac{1}{n} \left[\int_{\partial\Omega} H_{k-1} |\nabla u|^{k+1} \left(|\nabla u| - \langle x, \nu \rangle \right) \right]$$

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$$\binom{n}{k}^{-1} \frac{2}{k} \int_\Omega (-u) L[P]\, dx \le \frac{1}{n} \left[\int_{\partial\Omega} H_{k-1} |\nabla u|^{k+1} \left(|\nabla u| - \langle x, \nu \rangle \right) \right]$$

Work in progress...

- Obtain finer results for Serrin's overdetermined problem for the Hessian operators: doubling the exponent of the stability: δ instead of $\delta^{\frac{1}{2}}$, using $\delta = |||\nabla u| - R||_{L^2(\partial\Omega)}$. (see [Poggesi, *JAMPA*, 2025] for the case $k = 1$).

- Using the fundamental integral identity in order to obtain a stability result for the higher order SBT. Recall

Theorem [A. Ros, *Rev. Math. Iber.*, 1987]

The sphere is the only embedded compact hypersurface in the Euclidean space with H_k is constant, for some k.

Thank you for the attention!