mCost 2026
Milan

Isospectral quantum graphs

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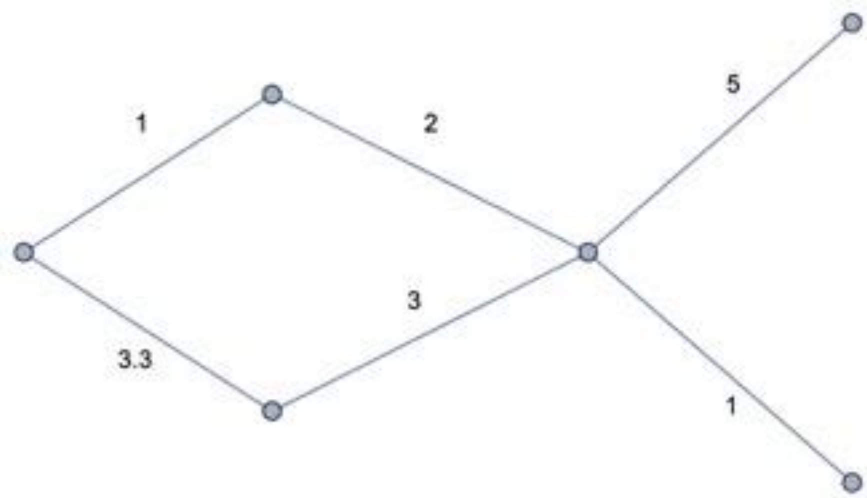
A weighted graph



The Laplace operator, L= $-\frac{\mathrm{d}^2}{\mathrm{d}x^2}$, is defined on each edge.

The solutions on each edge are given by $ae^{ikx} + be^{-ikx}$.

Boundary conditions

I use these boundary conditions at the vertices, V_m.

$$\begin{cases} f(x_i) = f(x_j) = f_0, \quad x_i, x_j \in V_m, \\ \sum\limits_{x_i \in V_m} \partial_n f(x_i) = \alpha f_0 \end{cases} \qquad \delta(\alpha)$$

The solutions are thus continuous at each vertex.
If $\alpha = 0$ we have the standard BC. I call the BC $\delta(\alpha)$.

There are more BC that are invariant under permutation of edges and have been classified by P. Exner et al.

Permutation invariant boundary conditions

These permutation invariant BCs are.

$$\begin{cases} \partial_n f(x_i) = \partial_n f(x_j) = \partial_n f_0, \quad x_i, x_j \in V_m, \\ \displaystyle\sum_{x_i \in V_m} f(x_i) = \beta\, \partial_n f_0 \end{cases}$$

β real - $\delta'_s(\beta)$.

$$\begin{cases} f(x_i) - f(x_j) = \gamma(\partial_n f(x_i) - \partial_n f(x_j)), \quad x_i, x_j \in V_m, \\ \displaystyle\sum_{x_i \in V_m} \partial_n f(x_i) = 0 \end{cases}$$

$\delta'(\gamma)$.

$$\begin{cases} \partial_n f(x_i) - \partial_n f(x_j) = \epsilon(f(x_i) - f(x_j)), \qquad x_i, x_j \in V_m, \\ \displaystyle\sum_{x_i \in V_m} f(x_i) = 0 \end{cases}$$

$\delta_p(\varepsilon)$.

I can also use Dirichlet boundary conditions at terminal vertices,
That is, the functions have value zero there.

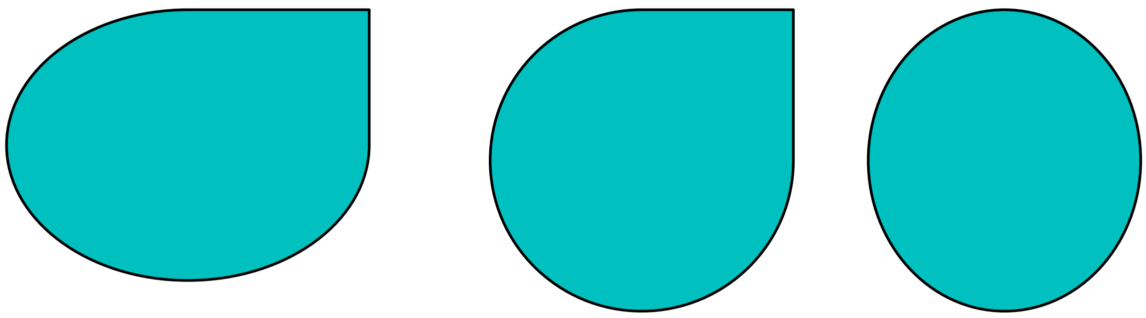
Usefulness of computer

Every compact graph has a secular determinant, $\Sigma(k)$, that determines the set of eigenvalues, the spectrum. I can compute $\Sigma(k)$ **exactly** for many boundary conditions.

I have tested against virtually all known results from literature.

Terence Tao has noted the usefulness of computers to find examples, that later can be proved by hand.

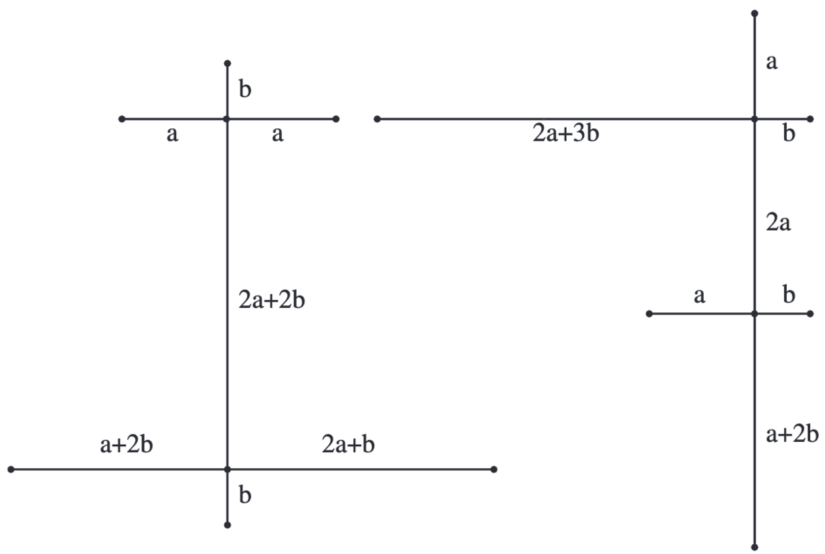
Can you hear the shape of a drum? M. Kac



Do some of these domains give the same spectrum for the Laplacian?
If the boundary if analytic?
Plenty of open questions.
3D

Isospectrality

Two graphs are isospectral if they have the same secular determinant.
Only a few examples were previously known.
People find such examples interesting and so do I.



Standard BC

B. Gutkin, U Smilansky. J. Phys. A: Math. Gen.34 2001
R Band, T Shapira, U Smilansky - Journal of Physics A: Mathematical and General, 39;45, 2006

Testing for isospectrality

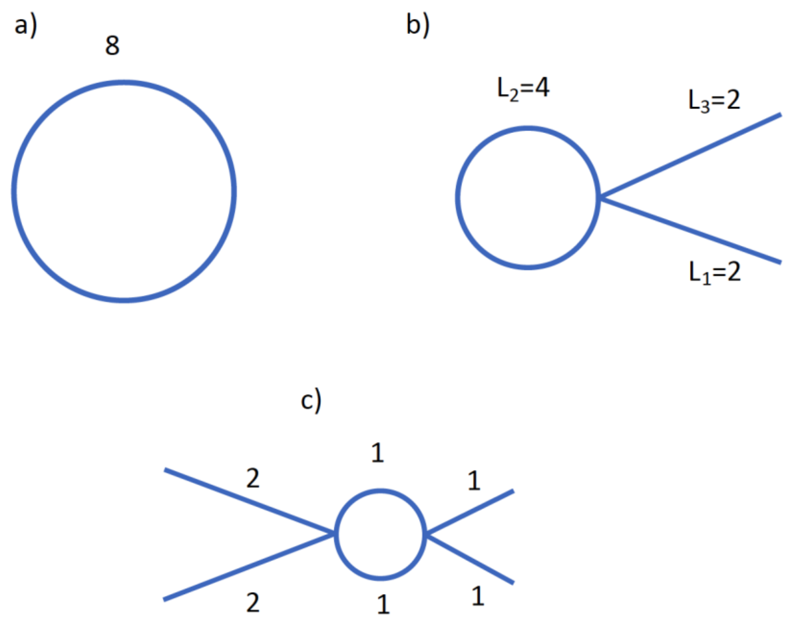
Below are isospectral sets among equilateral graphs with the indicated number of vertices. Found by exhaustive computer search. Standard and Neumann BC.

The fraction of isospectral graphs goes down with the number of vertices.

Vertices	Graphs	Pairs	Triplets	Sets of four	Trees
6	112	1	0	0	0
7	853	5	0	0	0
8	11117	39	3	0	0
9	261080	304	10	1	1
10	12005168				2
11	1018997864				5
12					6
13					37

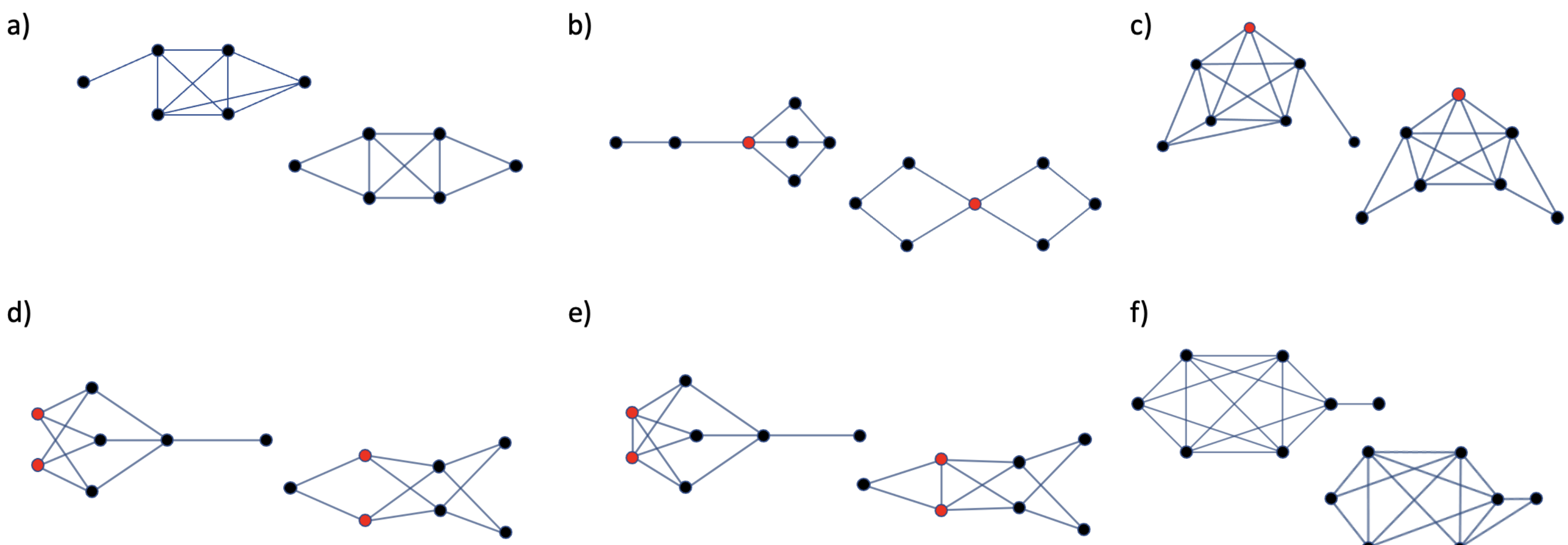
Isospectral partners of the loop

I find that the loop has two isospectral partners. Standard BC, $\delta(0)$.



Systematic search

Here are all six pairs of isospectral equilateral graphs with at most seven vertices. Standard BC, $\delta(0)$.



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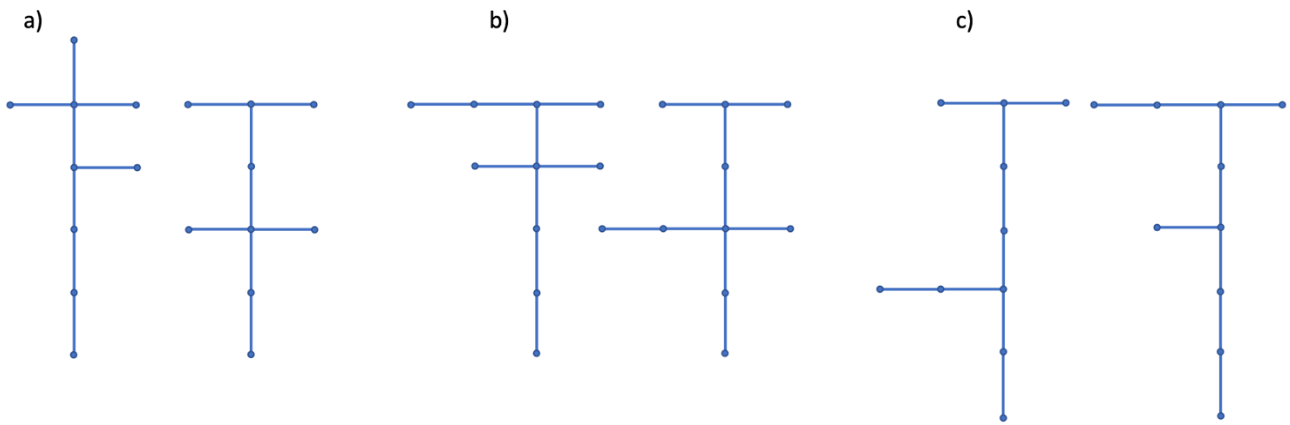
Isospectral trees

Here are all equilateral tree graphs with at most ten vertices.
Standard BC, $\delta(0)$. Neumann at terminal vertices.

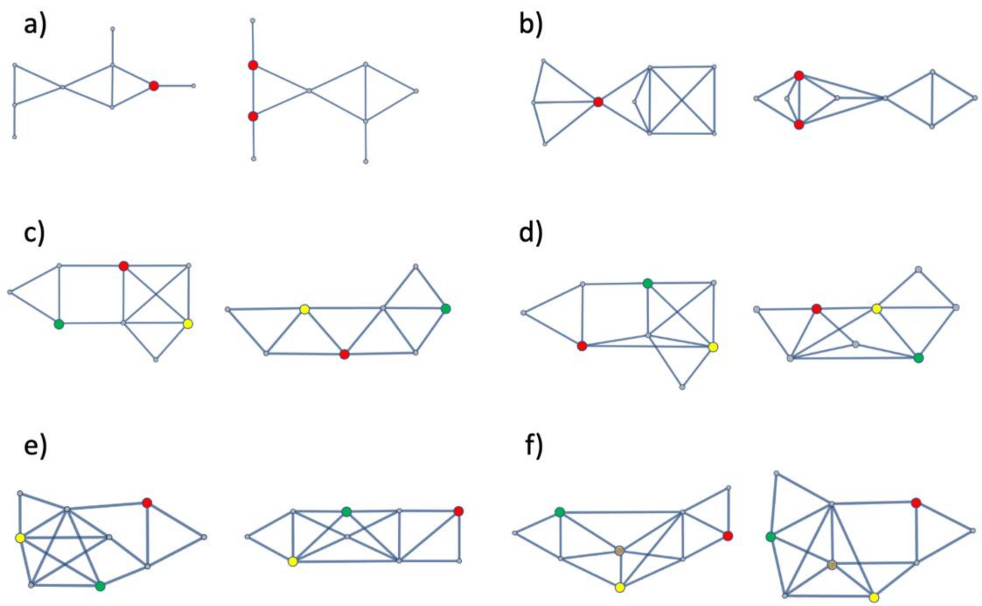


nine vertices ten vertices ten vertices

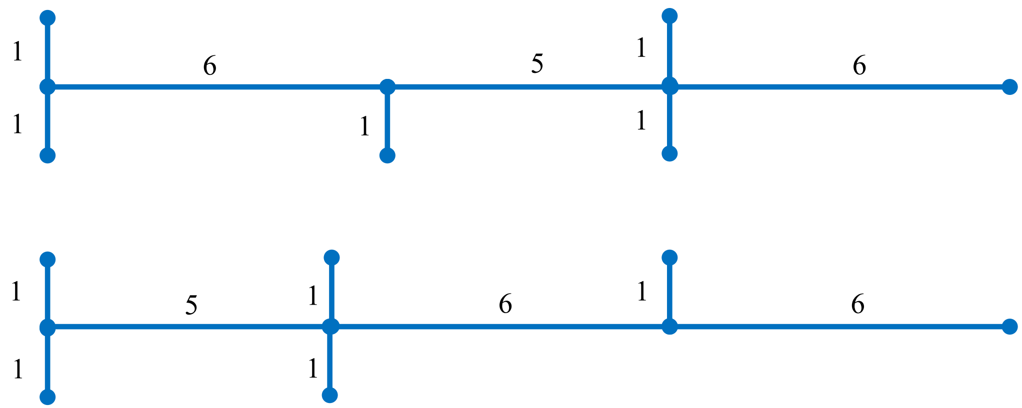
Let’s become more general.

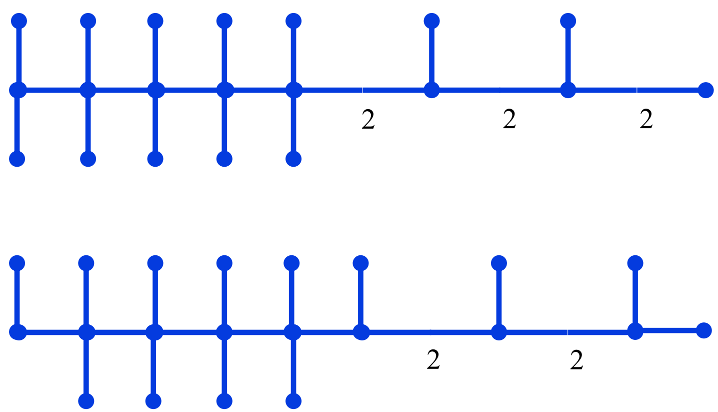
Here are all six pairs of isospectral equilateral graphs with at most eight vertices.

$\delta(\alpha)$ BC at internal vertices, Neumann at terminal vertices.



Let’s be general.





$\delta(\alpha)$ BC,

Neumann or Dirichlet at terminal vertices

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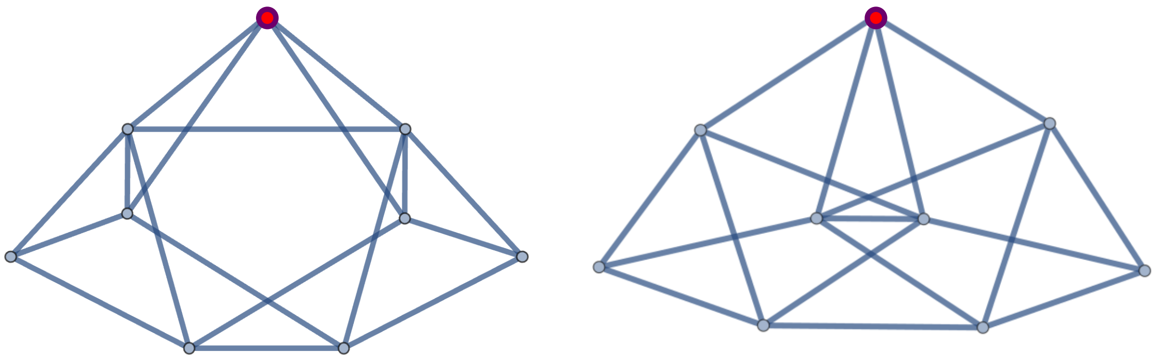
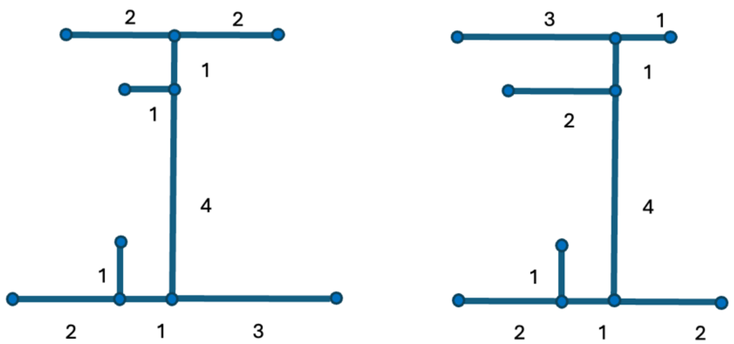
Let’s become even more general.



$\delta(\alpha)$, $\delta'_s(\beta)$, $\delta'(\gamma)$, $\delta_p(\varepsilon)$ BC.

A truly remarkable pair.

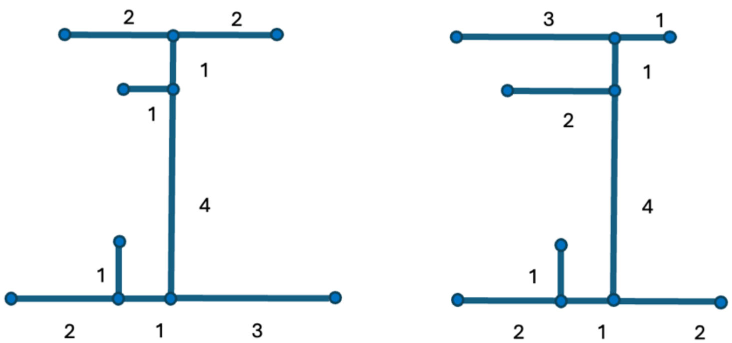
Let’s be very general.

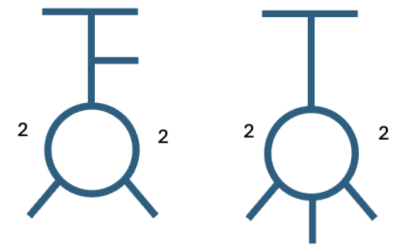


$\delta(\alpha)$, $\delta'_s(\beta)$, $\delta'(\gamma)$, $\delta_p(\varepsilon)$ BC,

Neumann or Dirichlet at terminal vertices.

Let’s be very general.





$\delta(\alpha)$, $\delta'_s(\beta)$, $\delta'(\gamma)$, $\delta_p(\varepsilon)$ BC.

Neumann or Dirichlet at terminal vertices.

Titchmarsh-Weyl M-function, Dirichlet to Neumann map

$$M^{\Gamma}(k)u(k,\partial\Gamma) = u'(k,\partial\Gamma)$$

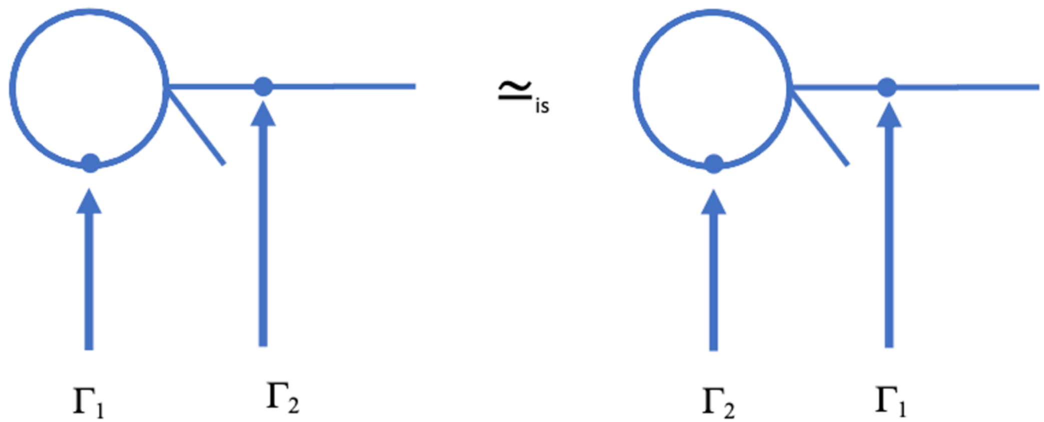
Maps the value of function to its sum of outwards derivatives at a contact vertex, $\partial\Gamma$, or a set of contact vertices.

If two vertices in one graph have the same M-function one can attach a graph to either one of them and get isospectral graphs.

I wrote a program to find M-functions fast.

Berkolaiko, Kuchment, Introduction to quantum graphs, volume 186. American Mathematical Soc., 2013.
Pavel Kurasov and Jacob Muller. On isospectral metric graphs. arxiv.org/abs/2112.04230

Graphs with vertices having the same M-function



Γ_1 and Γ_2 can have any (self-adjoint) boundary conditions.

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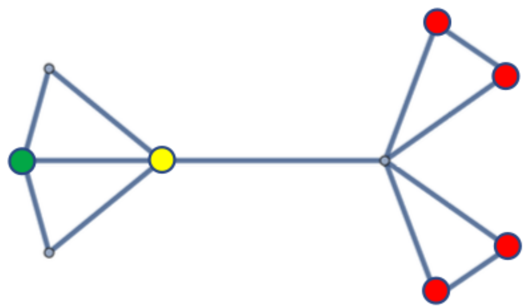
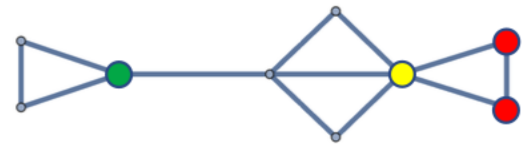
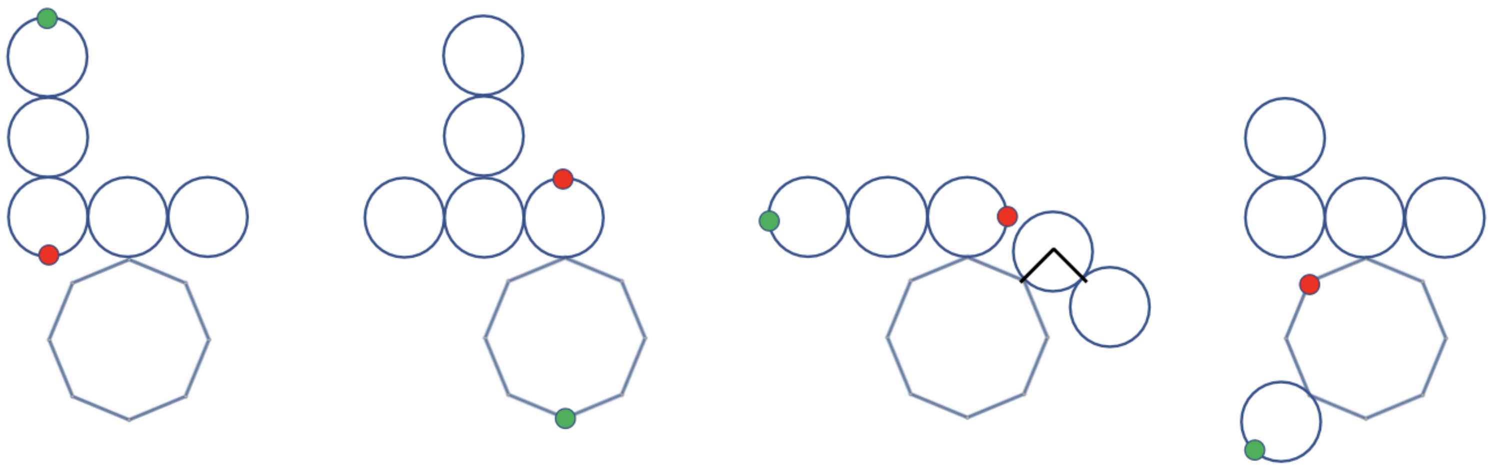
Isospectral graphs with vertices having the same M-function





All vertices with the same colour have the same M-function.
Standard BC.

Isospectral graphs with vertices having the same M-function



21 sets of vertices with the same M-function.

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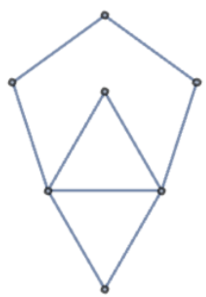
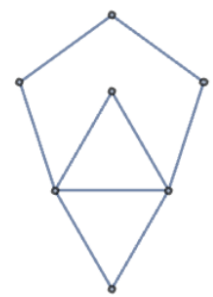
Isospectral graphs under different boundary conditions

$\{\Gamma_1, b\}$ isospectral to $\{\Gamma_2, b\}$

$\{\Gamma, b_1\}$ isospectral to $\{\Gamma, b_2\}$



$\delta(0)$



$\delta_p\,(0)$

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I have searched more than 100 million graphs.
Universally isospectral graphs are exceedingly rare.

Python, Mathematica.

Vibe-coding in the cloud (Google Cloud) to get candidate pairs.

Mathmatica to test candidates, locally on my laptop.

Code speed-up typically 10 000 times, often more.
In the old days I could only test thousands of graphs per night.

In the future I will probably go into vibe-proving. So far no success. Then Lean.

Outlook

Start to formulate conjectures and prove them, using AI or not. Confirm using Lean. (I have some observations.)

Two particles on graphs, will it break isospectrality? I doubt it in general.

Check for graphs being even more isospectral, using the more tokens (more compute time) I now have available.

Time-dependence.

Other (diffusion) operators.

Superlattices of boundary conditions.

Open source

The eigenvalue program along with all tests is available at GitHub:
https://github.com/meapistol/Spectra-of-graphs

A manuscript is available at: https://arxiv.org/abs/2104.12885
The manuscript is continuously updated.

GitHub contains notebooks to generate all the isospectral graphs which are included in the manuscript (and many more).