Multi-cluster diffusion limited aggregation

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TECHNION
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Milano - Multiscale stochastics, patterns, and anlaysis of combinatorial environments

Joint work with Noam Berger (TUM), Dominik Schmid (University of Augsburg) and Daniel Sharon (Technion)

Single-particle with single aggregate - DLA in $\mathbb{Z}^d$

- $A_0 = \{0\}$ and $A_{n+1} = A_n \cup \{a_{n+1}\}$, where $a_{n+1} \sim \mathcal{H}_{\partial A_n}(\cdot)$. $\partial A = \{y \in \mathbb{Z}^2 \setminus A : \exists x \in A, y \sim x\}$.
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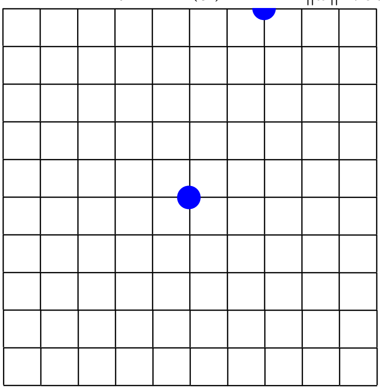
- 
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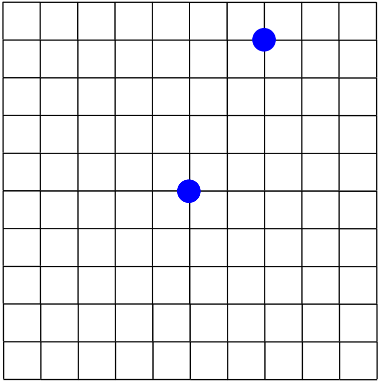

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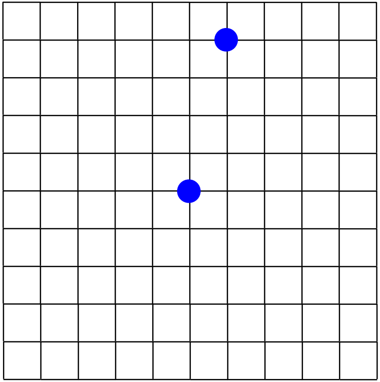

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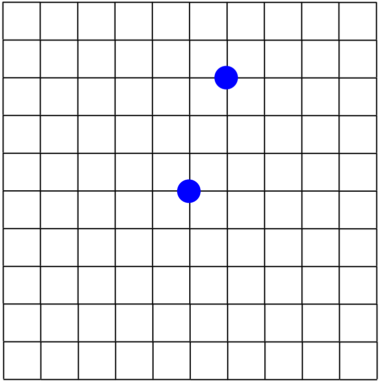
- 
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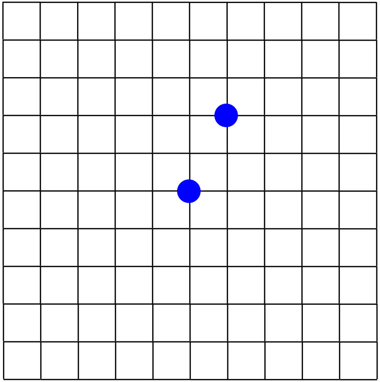

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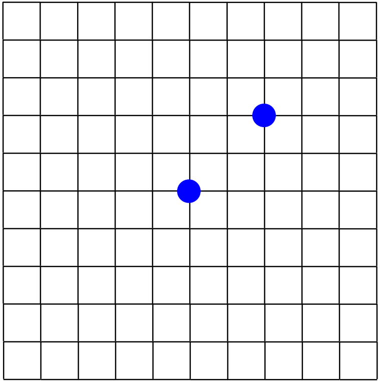

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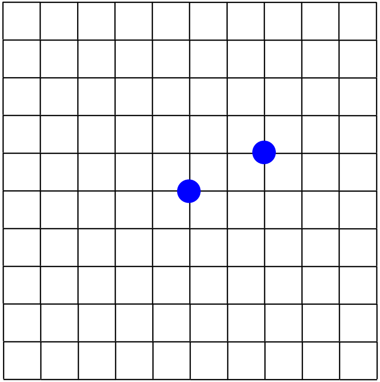

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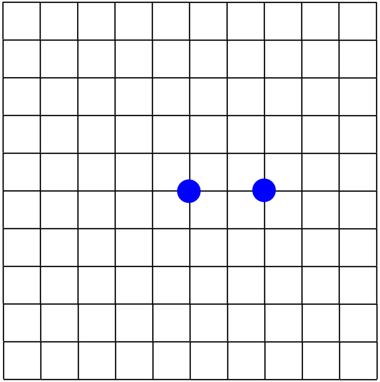
- 
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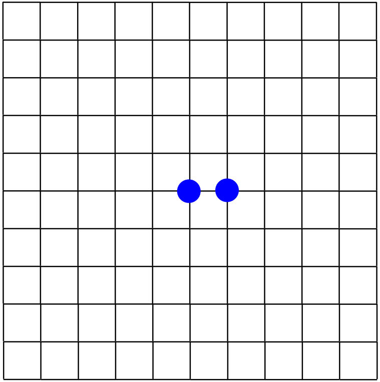

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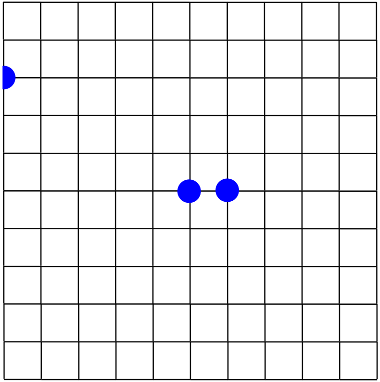

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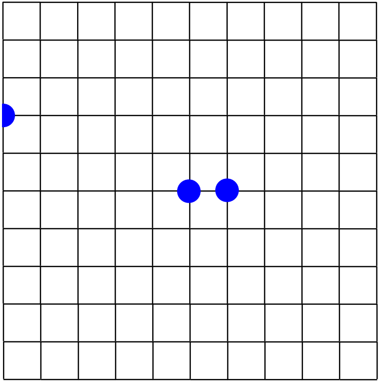

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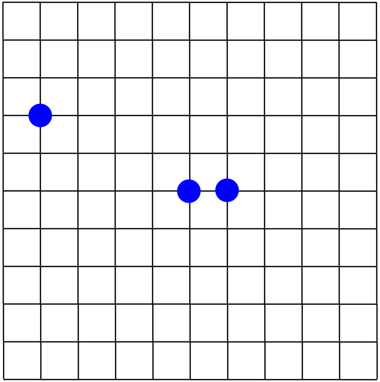

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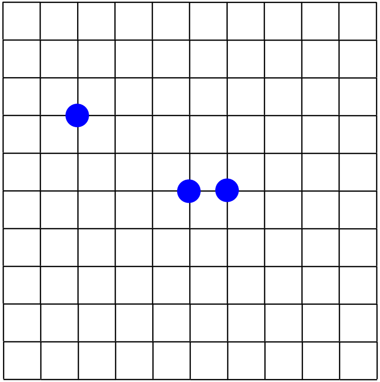

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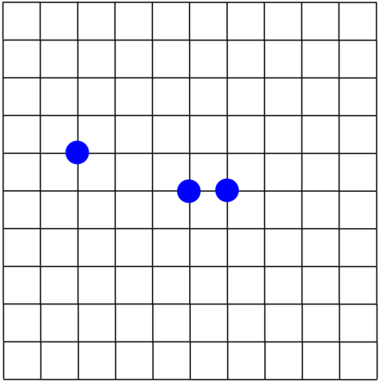

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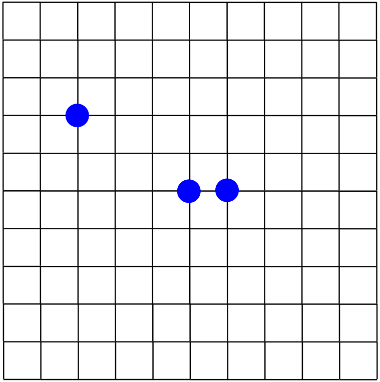

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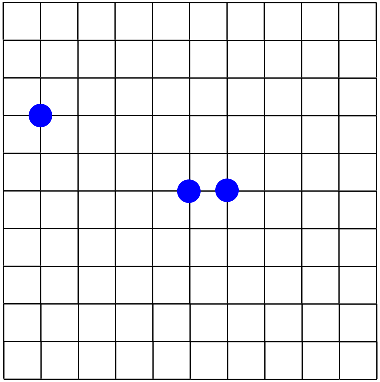

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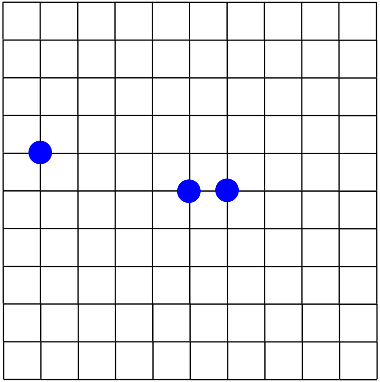

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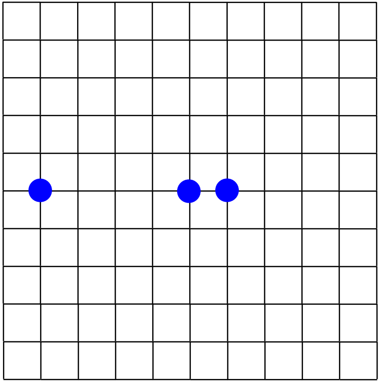

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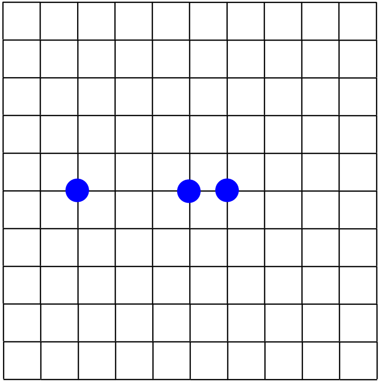

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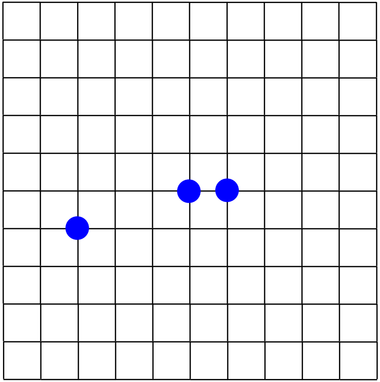

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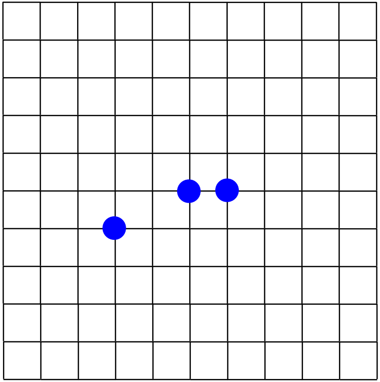

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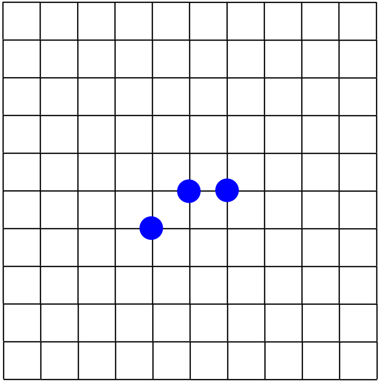

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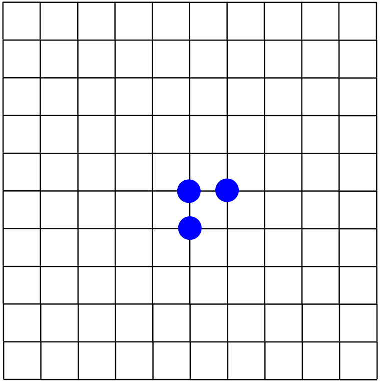

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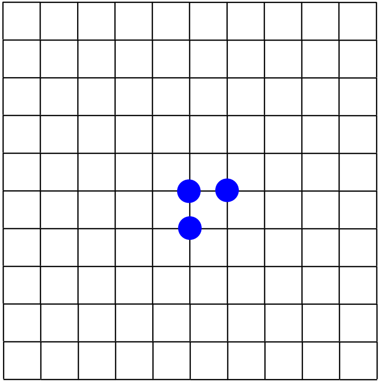
- 
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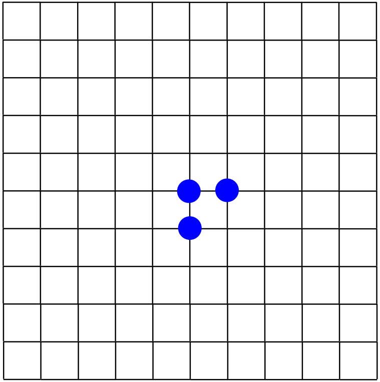
- 
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- Multi-Particle DLA was defined by Rosenstock, Marquardt 80 and Voss 84.
- Fix $\lambda > 0$. Start with an aggregate $\mathscr{M}_0 = 0$, and in every $x \in \mathbb{Z}^d$, place Poi(λ) i.i.d number of particles.
- Particles preform ind continuous time random walk.
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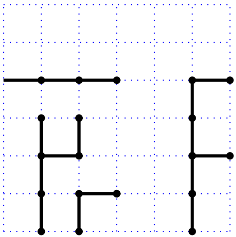

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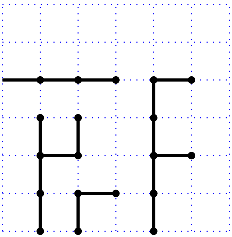

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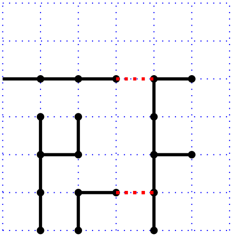

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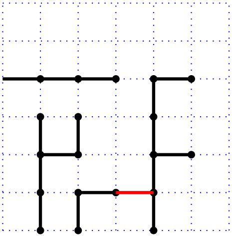

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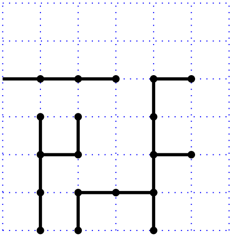

Meakin's Conjecture

- "More extensive simulations indicate that a fractal dimensionality of ≈ 1.75 will be obtained for all versions of the model in which the diffusion coefficient of small clusters is equal to or greater than the diffusion coefficient for larger clusters." i.e. $\alpha \geq 0$.
- CC simulation with $p = 0.05$, $\alpha = 0$.
  All clusters move at rate 1.
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- Gidi Amir: Does CC admit an infinite cluster in finite time?

Theorem 5 (Berger, P, Schmid, Sharon 26)

For all $d \geq 1$, $\alpha \geq 0$, $t \geq 0$, all clusters are finite a.s. (no condensation).

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For all $d > 1$, $\alpha < -1 - \frac{2}{d}$, immediate blowup. Easy for $p = 1, \alpha < -1$ by coupling to Bernoulli percolation.

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For all $d > 1$, find $\alpha_c(d)$ such that $\forall \alpha > \alpha_c(d)$, finite clusters a.s. $\forall t > 0$. 2 phase transitions?

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Results and Open Problems - $d = 1, p \in (0, 1)$

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Proof sketch - $\alpha \geq 0$, Upper bound

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Fix $\epsilon > 0$, and consider 4 specific clusters:

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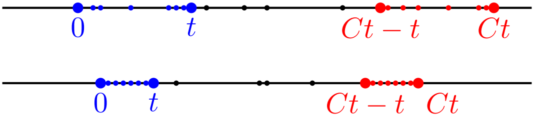

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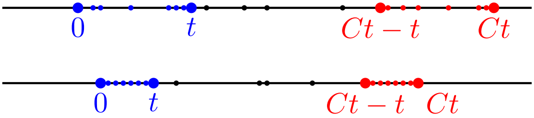

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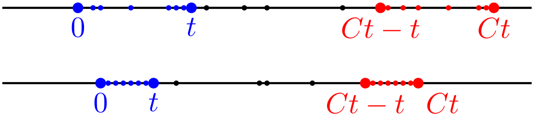

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Theorem 10 (Berger, P, Sharon 25)

Let $d = 1, p = 1, \alpha \in [-1, 0)$. For small enough t, there is no infinite cluster a.s, and for large enough t there exists an infinite cluster a.s.

- Most interesting regime so far - we have $t_c(\alpha)$ depending on $\alpha \in [-1, 0)$.

Open Problem 3

Find $t_c(\alpha)$, and show $t_c(1) = 1$.

- Proofs give bounds to $t_c(\alpha)$ which can be calculated.

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Importance of Starting Configuration

Theorem 11

Let $d > 1, \alpha < 0, p > 0$. There exists a translation invariant starting configuration that explodes immediately a.s.

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Create a starting configuration inductively:

- Start with $\mathbb{Z}^d$.
- Create a hollow $\ell \times \ell \times ... \times \ell \times 3$ rectangle tiling by opening edges
- Shift the tiling randomly.
- Treat each rectangle as a node and repeat the process.

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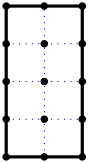

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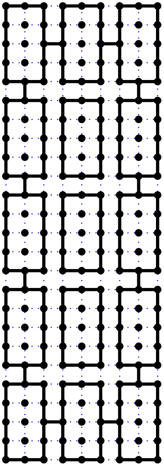

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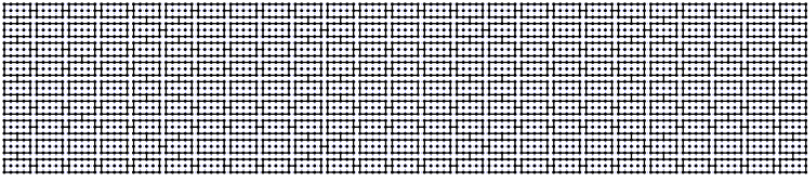

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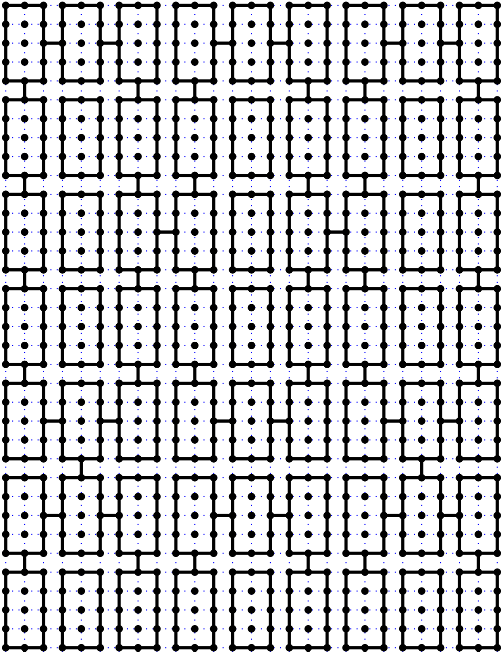

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The configuration is valid:

1. Translation invariant - Immediate from the random shift.
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- $\mathscr{C}_x$ needs to connect to the outer rectangle infinitely many times.
- Consider connection along the largest face at renormalization layer k.
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- In total: effective outwards rate is $\geq \frac{1}{2d}\left(\frac{\left(\ell^{d-1}\right)^{-\alpha}}{3}\right)^k > 2^k$ for large enough ℓ.
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Non exploding starting configuration - on the board

Thank you for your attention!

Adulations?

Questions?

Remarks?

Complaints?