



# On the existence of optimal shapes in architecture

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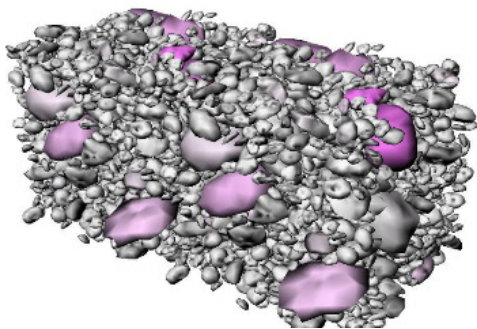
Infimum for Lipschitz case

Minimum for non-Lipschitz case

## **Open problems and perspectives**

## Nature complexity and their models

### Porous materials



# Traffic noise absorbing wall

**“Fractal wall” TM**, porous material is the cement-wood (acoustic absorbent),  
Patent Ecole Polytechnique-Colas, Canadian and US patent



# Acoustic anechoic chambers

Test anechoic chamber

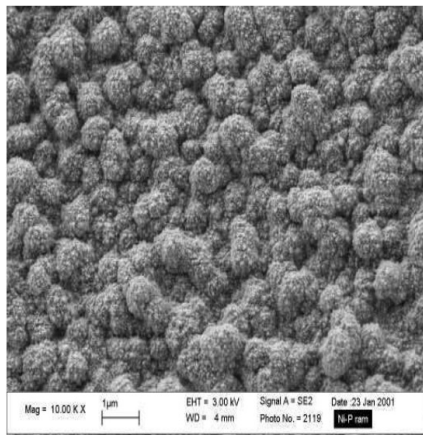


Microsoft anechoic chamber -20db noise level,  
the quietest place on earth

Test semi-anechoic chamber

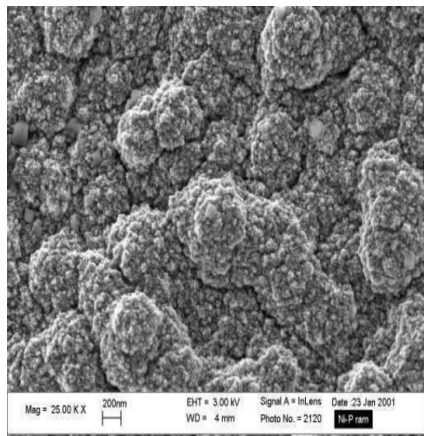


## Irregularity of boundaries



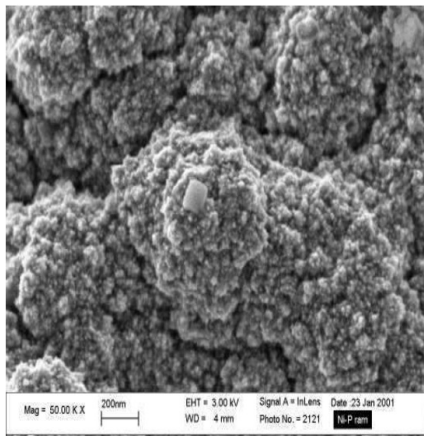
$1 \mu\text{m}$

## Irregularity of boundaries



←→  
 $1 \mu m$

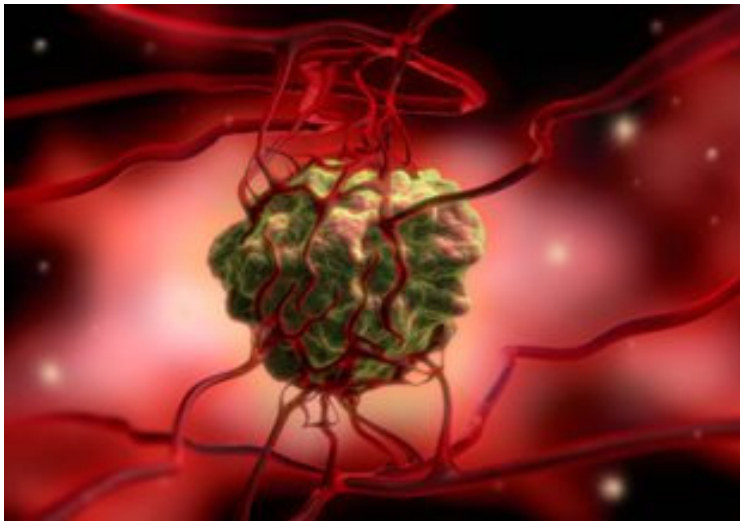
## Irregularity of boundaries



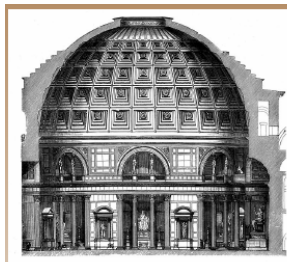
1  $\mu m$

## Irregularity of boundaries

Angiogenesis of cancerous tumours



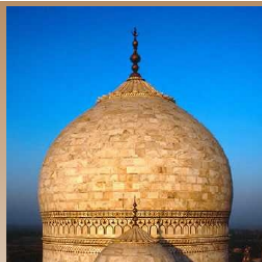
## Examples of the roof structures



Spherical dome in Rome, Italy  
(125)



Dome in Nagorny Karabakh (1201)



Dome of Taj Mahal, India (1632)



African domes (paraboloid of revolution)



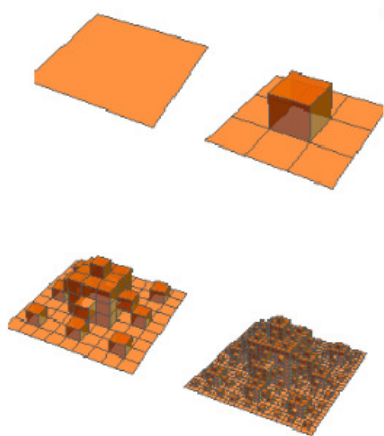
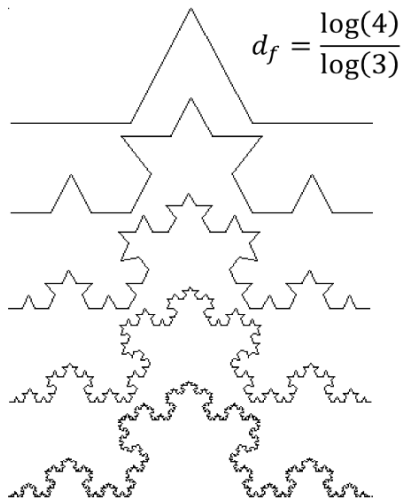
Traditional buildings of Cameroon



## The precipitations: some troubles from snow



## Examples of self-similar fractal boundaries



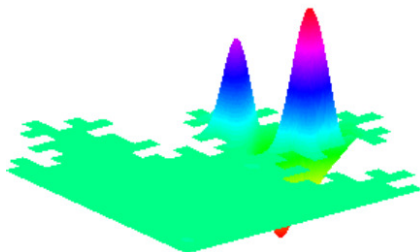
$$2 < d = \frac{\log(13)}{\log(3)} \approx 2.33 < 3 \quad (\text{Wikipedia})$$

## Optimality of fractal models in applications

Physically, models involving fractals are the most

- stable,
- conductive,
- dissipative,
- absorbing structures.

Works of B. Sapoval, M. Filoche, D. Grebenkov and their co-authors:



Localization (weak) of eigenfunctions

## “Fractal” project: existence of optimal shapes

- **In acoustics** (mixed boundary conditions Dirichlet/Neumann/Robin)
  - F. Magoulès, P.T.K. Ngyuen, P. Omnes, A. Rozanova-Pierrat, *Optimal absorbtion of acoustic waves by a boundary*. SIAM J. Control Optim. (2021).
  - M. Hinz, A. Rozanova-Pierrat, A. Teplyaev, *Non-Lipschitz uniform domain shape optimization in linear acoustics*. SIAM J. Control Optim. (2021).
- **In architecture** (non-homogeneous Dirichlet and Neumann conditions)
  - M. Hinz, F. Magoulès, A. Rozanova-Pierrat, M. Rynkovskaya, A. Teplyaev, *On the existence of optimal shapes in architecture*. Appl. Math. Model., (2021).
- **In heat exchanges** (transmission problem)
  - G. Claret, A. Rozanova-Pierrat, *Existence of optimal shapes for heat diffusions across irregular interfaces*, accepted in AMS, CONM book series (2024).
- **In the elliptic framework with Robin type condition**
  - M. Hinz, A. Rozanova-Pierrat, A. Teplyaev, *Boundary value problems on Non-Lipschitz uniform domains: Stability, Compactness and the Existence of optimal shapes*. Asymptotic Analysis, (2023).

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## Open problems and perspectives

## Notations

- The full contraction

$$A : B = \operatorname{tr} A^t B = \sum_{i,j=1}^N a_{ij} b_{ij}$$

of two matrices  $A = (a_{ij})_{1 \leq i,j \leq N}$  and  $B = (b_{ij})_{1 \leq i,j \leq N}$  from  $\mathcal{M}_N$ .

- $\alpha > \mathbf{o}, \beta > \mathbf{o}$ , let  $\mathcal{M}_N^S(\alpha, \beta)$  denote the subspace of all invertible  $M \in \mathcal{M}_N^S$  such that

$$M\xi \cdot \xi \geq \alpha|\xi|^2 \text{ and } M^{-1}\xi \cdot \xi \geq \beta|\xi|^2 \quad \forall \xi \in \mathbb{R}^N$$

- $\Omega \subset \mathbb{R}^N, \mathbf{v} \in H^1(\Omega)^N$ , we denote the symmetric part of its gradient by

$$\mathbf{e}(\mathbf{v}) = \frac{1}{2} (\nabla \mathbf{v} + (\nabla \mathbf{v})^t) \in L^2(\Omega, \mathcal{M}_N^S)$$

- $T = (T_{ij})_{1 \leq i,j \leq N}$  of  $H^1(\Omega)^{N \times N}$  we define

$$\operatorname{div} T = \left( \sum_{j=1}^N \partial_{x_j} T_{1j}, \dots, \sum_{j=1}^N \partial_{x_j} T_{Nj} \right) \in L^2(\Omega)^N$$

## Problem

Let  $\mathbf{A} \in L^\infty(\Omega, \mathcal{M}_N^S(\alpha, \beta))$ ,  $\sigma(\mathbf{u}) = \mathbf{A}e(\mathbf{u})$ ,  $\mathbf{u} \in H^1(\Omega)^N$ .

$$\begin{cases} -\operatorname{div} \sigma(\mathbf{u}) &= \mathbf{f} & \text{in } \Omega, \\ \mathbf{u} &= \mathbf{0} & \text{on } \Gamma_{\text{Dir}}, \\ \sigma(\mathbf{u}) \cdot \mathbf{n} &= \mathbf{g} & \text{on } \Gamma_{\text{Neu}}. \end{cases} \quad (1)$$

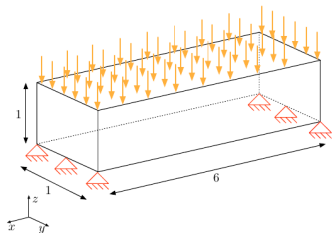
- The domain  $\Omega$  is 'clamped' at  $\Gamma_{\text{Dir}}$ .
- $\Gamma_{\text{Neu}}$  with  $\mathbf{g} \neq \mathbf{0}$  is subjected to traction represented by the force field  $\mathbf{g}$ .
- The vector field  $\mathbf{f}$  represents a force field experienced inside  $\Omega$ .
- $\mathbf{u}$  is the (unknown) displacement vector field.
- $e(\mathbf{u})$  is the strain tensor, and  $\sigma(\mathbf{u})$  is the stress tensor, determined by the given coefficient  $\mathbf{A}$ , which is minus the elasticity tensor in Hooke's law.
- For isotropic materials  $\sigma(\mathbf{v}) = \lambda(\operatorname{tr} e(\mathbf{v}))I + 2\mu e(\mathbf{v})$ , where  $\mu > \mathbf{0}$  and  $\lambda > -2\mu/N$  are the Lamé coefficients of the material.

## Known results: numerical shape optimization $\min_{\Omega \in U_{ad}} J(\Omega)$

- **C. Dapogny, A. Faure, G. Michailidis, G. Allaire, A. Couvelas, and R. Estevez**, Geometric constraints for shape and topology optimization in architectural design, *Computational Mechanics*, 59 (2017), pp. 933–965.

$$J(\Omega) = \int_{\Omega} \mathbf{A}e(\mathbf{u}) : e(\mathbf{u}) dx = \int_{\Omega} \mathbf{f} \cdot \mathbf{u} dx + \int_{\Gamma_N} \mathbf{g} \cdot \mathbf{u} ds$$

$$U_{ad} = \{ \Omega \subset \mathbb{R}^N \text{ is a smooth bounded domain, } \Gamma_D \cup \Gamma_N \subset \partial\Omega \}$$



## Open problem before A.R-P-2021

**Does exist an optimal shape of an architecture structure under the loads?**

To define a compact class of domains under weak boundary measure convergence.

M. Hinz, F. Magoulès, A. R-P, M. Rynkovskaya, A. Teplyaev, *On the existence of optimal shapes in architecture*. Appl. Math. Model., (2021).

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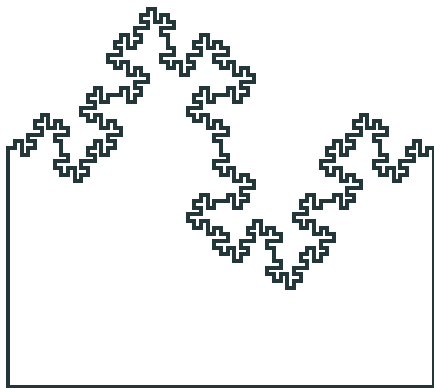
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Minimum for non-Lipschitz case

## Open problems and perspectives

## Main difficulties for non-Lipschitz boundary value problems

A fractal and Lipschitz curve boundary  $\partial\Omega$  of dimension  $d \geq n - 1$



No elliptic regularity of solutions:  $u \notin H^2(\Omega)$

No classical normal derivative  $\nabla u \cdot \vec{n}$  ( $\nexists \vec{n}$ )

K. Nyström, 1996, von Koch's snowflake, NTA domains; P. Grisvard, 1974, Lipschitz non-convexe

## Sobolev extension domains

### Definition

A domain  $\Omega \subset \mathbb{R}^n$  is called a  **$H^1$ -extension domain** if there exists a bounded linear extension operator  $E_\Omega : H^1(\Omega) \rightarrow H^1(\mathbb{R}^n)$ :

$$\forall u \in H^1(\Omega) \quad \exists v = E_\Omega u \in H^1(\mathbb{R}^n) \text{ with } v|_\Omega = u \text{ and } C(\Omega) > 0 :$$

$$\|v\|_{H^1(\mathbb{R}^n)} \leq C \|u\|_{H^1(\Omega)}.$$

**Jones [1981]:** If  $\Omega$  is an uniform (or  $(\varepsilon, \infty)$ -) domain, then it is Sobolev extension domain.

**Hajlasz, Koskela and Tuominen [2008]:**  $\Omega \subset \mathbb{R}^n$  is a  $H^1$ -extension domain  $\iff \Omega$  is an  $n$ -set and  $H^1(\Omega) = C^{1,2}(\Omega)$  (space of the fractional sharp maximal functions) with norms' equivalence.

## Locally uniform or $(\varepsilon, \delta)$ -domains ( $\varepsilon > 0, 0 < \delta \leq \infty$ )

### Definition

An open connected subset  $\Omega$  of  $\mathbb{R}^n$  is an  $(\varepsilon, \delta)$ -domain,

if whenever  $x, y \in \Omega$  and  $|x - y| < \delta$ , **(thus locally)**

there is a rectifiable arc  $\gamma \subset \Omega$  with length  $\ell(\gamma)$  joining  $x$  to  $y$  and satisfying

1.  $\ell(\gamma) \leq \frac{|x-y|}{\varepsilon}$  **(uniformly locally quasiconvex)** and
2.  $d(z, \partial\Omega) \geq \varepsilon|x - z| \frac{|y-z|}{|x-y|}$  for  $z \in \gamma$ .

### Theorem ( $n = 2$ , Jones [1981])

A bounded and finitely connected domain  $\Omega$  is  $(\varepsilon, \infty)$ -domain  $\iff$  its boundary consists of a finite number of points and quasicircles.

## $n$ -sets or measure density condition

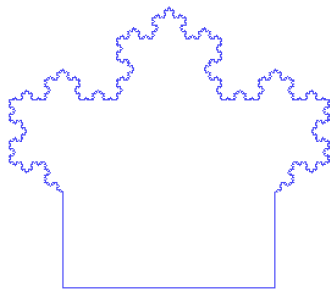
- All  $(\varepsilon, \delta)$ -domains in  $\mathbb{R}^n$  are  $n$ -sets ( $d$ -set with  $d = n$ ):

$$\exists c > 0 \quad \forall x \in \overline{\Omega}, \forall r \in ]0, \delta[ \cap ]0, 1] \quad \lambda(B_r(x) \cap \Omega) \geq C\lambda(B_r(x)) = cr^n,$$

where  $\lambda$  denotes the Lebesgue measure in  $\mathbb{R}^n$ .

- An  $n$ -set  $\Omega$  cannot be “thin” close to its boundary  $\partial\Omega$ , since it contains a non-trivial ball in its neighborhood.

## Examples of extension domains



Extension domain



Not an extension domain

# Trace operator [M. Biegert, 2009]: boundary measure free framework on extension domains

## Definition

$H^1$ -extension domain  $\Omega$  is called  **$H^1$ -admissible** if its boundary  $\partial\Omega$  has positive capacity.

# Trace operator [M. Biegert, 2009]: boundary measure free framework on extension domains

## Definition

$H^1$ -extension domain  $\Omega$  is called  **$H^1$ -admissible** if its boundary  $\partial\Omega$  has positive capacity.

## Proposition

For a  $H^1$ -admissible domain  $\Omega$  of  $\mathbb{R}^n$ , given  $u \in H^1(\Omega)$ , let

$$\text{Tr } u := (E_\Omega u)^\sim|_{\partial\Omega}$$

be the restriction of any quasi continuous representative  $(E_\Omega u)^\sim$  of  $E_\Omega u$ . Then the **trace operator**

$$\text{Tr} : H^1(\Omega) \rightarrow \mathcal{B}(\partial\Omega)$$

is a well-defined linear surjection. Consequently, q. e.

$$x \in \partial\Omega \quad \text{Tr } u(x) = \lim_{r \rightarrow 0} \frac{1}{\lambda^n(\Omega \cap B_r(x))} \int_{\Omega \cap B_r(x)} u(y) dy.$$

## Trace theorem (“boundary measure free”)

Let  $\Omega \subset \mathbb{R}^n$  be an  $H^1$ -admissible bounded domain.

$$H^1(\Omega) = H_0^1(\Omega) \oplus V_1(\Omega), \quad V_1(\Omega) = \{u \in H^1(\Omega) \mid -\Delta u + u = \mathbf{0} \text{ weakly}\}$$

- (i) The space  $H_0^1(\Omega) := \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{H^1(\Omega)}}$  is the kernel of  $\text{Tr}$ , that is,  $H_0^1(\Omega) = \ker \text{Tr}$ .
- (ii) Endowed with the norm

$$\|f\|_{\mathcal{B}(\partial\Omega)} := \min\{\|v\|_{H^1(\Omega)} \mid v \in H^1(\Omega) \text{ and } \text{Tr } v = f\}, \quad (2)$$

the space  $\mathcal{B}(\partial\Omega)$  is a Hilbert space.

- (iii)  $\|\text{Tr}_i\|_{\mathcal{L}(H^1(\Omega), \mathcal{B}(\partial\Omega))} = 1$ .

Its restriction  $\text{tr}|_{V_1(\Omega)} : V_1(\Omega) \rightarrow \mathcal{B}(\partial\Omega)$  to  $V_1(\Omega)$  is an isometry and onto.

## Boundary measure framework of admissible domains $(\Omega, \Gamma)$

Let  $\Omega$  be a bounded  $H^1$ -extension domain of  $\mathbb{R}^n$  and  $\Gamma \subset \bar{\Omega}$  is such that

- $\Gamma$  has positive capacity
- $\Gamma$  is a compact support  $\Gamma = \text{supp } \mu$  for a positive Borel measure  $\mu$  on  $\mathbb{R}^n$  satisfying for  $d > 0$ ,  $d \in ]n - 2, n[$  the **upper  $d$ -regular** condition: there is a constant  $c_d > 0$  such that

$$\mu(B_r(x)) \leq c_d r^d, \quad x \in \Gamma, \quad 0 < r \leq 1. \quad (3)$$

- $A \subset \Gamma, \quad \text{Cap}(A) = 0 \Rightarrow \mu(A) = 0.$

( $\implies \dim_H \Gamma \geq d$ )

**Examples:** union of different  $d$ -sets, multifractals, ...

$\Gamma = \partial\Omega$  is a standart case.

## Examples of boundaries and boundary measures

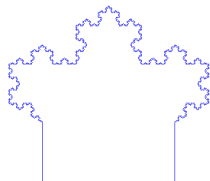
Domains with boundaries  $\partial\Omega$  as

- $d$ -sets:  $\dim_H \partial\Omega = d > 0$   
 $\exists c_1, c_2 > 0,$

$$c_1 r^d \leq \mu(\partial\Omega \cap \overline{B_r(x)}) \leq c_2 r^d, \quad \text{for } \forall x \in \partial\Omega, 0 < r \leq 1,$$

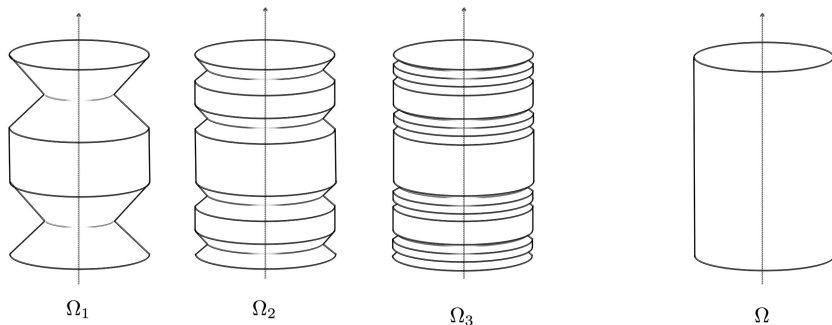
- Lipschitz and more regular boundaries
- bounded dimension boundaries

$$n - 2 < \dim_H \partial\Omega \leq n$$



## Examples of boundaries and boundary measures

upper  $d$ -regular measure with a regular support<sup>a</sup>



<sup>a</sup>N. Alami, E. Lucyszyn, R. Pain Dit Hermier, ARP, *Parametric shape optimization for the convected Helmholtz equation with a generalized Myers boundary condition* .

<https://hal.science/hal-05007470>

## Definition of the trace operator [A. Jonnson, 2009; M. Biegert, 2009]

### Definition

For a Sobolev extension domain  $\Omega$  of  $\mathbb{R}^n$  with  $\text{supp } \mu = \partial\Omega$  (for an upper regular Borel measure  $\mu$ ),

the trace operator  $\text{Tr} : H^1(\Omega) \rightarrow L^2(\partial\Omega, \mu)$  is defined  $\mu$ -a.e. by

$$x \in \partial\Omega \quad \text{Tr } u(x) = \lim_{r \rightarrow 0} \frac{1}{\lambda^n(\Omega \cap B_r(x))} \int_{\Omega \cap B_r(x)} u(y) dy.$$

Properties of  $\partial\Omega$ ,  $\partial\Omega = \text{supp } \mu$  are important to characterize the norm of  $B(\partial\Omega) := \text{Tr}(H^1(\Omega))$ :

$$H^{\frac{1}{2}}(\partial\Omega), \quad B_{1-\frac{n-d}{2}}^{2,2}(\partial\Omega), \quad B_1^{2,2}(\partial\Omega), \quad \dots$$

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d-sets, H. Wallin 1991

A. Jonnson 1997

# Trace theorem on boundaries given by upper $d$ -regular measures $\mu$

Let  $(\Omega, \partial\Omega)$  be admissible. Then

- (i)  $\text{Tr} : H^1(\Omega) \rightarrow L^2(\partial\Omega, \mu)$  is **compact operator** and  $\exists c_{\text{Tr}}(n, \Omega, d, c_d) > 0$ , s. t.  
 $\|\text{Tr} f\|_{L^2(\partial\Omega, \mu)} \leq c_{\text{Tr}} \|f\|_{H^1(\Omega)}, \quad f \in H^1(\Omega).$
- (ii)  $B(\partial\Omega) := \text{Tr}(H^1(\Omega))$  is a Hilbert space, **compact and dense** in  $L^2(\partial\Omega, \mu)$

$$\|\varphi\|_{B(\partial\Omega)} := \min\{\|g\|_{H^1(\Omega)} \mid \varphi = \text{Tr } g\}.$$

(iii) **the Gelfand triple**

$$B(\partial\Omega) \hookrightarrow L^2(\partial\Omega, \mu) = (L^2(\partial\Omega, \mu))' \hookrightarrow B'(\partial\Omega), \quad B''(\partial\Omega) = B(\partial\Omega).$$

## Green formula

$$H(\operatorname{div}, \Omega) = \{\mathbf{u} = (u_1, \dots, u_N)^t : \Omega \rightarrow \mathbb{R}^N \mid u_1, \dots, u_N \in L^2(\Omega), \operatorname{div} \mathbf{u} \in L^2(\Omega)\},$$

it is a Hilbert space corresponding to the norm

$$\|\mathbf{u}\|_{\operatorname{div}, \Omega} = \left( \|\mathbf{u}\|_{L^2(\Omega)^N}^2 + \|\operatorname{div} \mathbf{u}\|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}}.$$

### Proposition

Let  $(\Omega, \mu) \subset \mathbb{R}^N$  be a bounded  $H^1$ -admissible domain. For all  $\mathbf{u} \in H(\operatorname{div}, \Omega)$ ,  $w \in H^1(\Omega)$   $\mathbf{u} \cdot \mathbf{n} \in \mathcal{B}'(\partial\Omega)$  is defined by the identity

$$\langle \mathbf{u} \cdot \mathbf{n}, \operatorname{Tr} w \rangle_{\mathcal{B}'(\partial\Omega), \mathcal{B}(\partial\Omega)} = \int_{\Omega} \mathbf{u} \cdot \nabla w \, dx - \int_{\Omega} (\operatorname{div} \mathbf{u}) w \, dx.$$

For all  $T \in H(\operatorname{div}, \Omega)^N$  we define  $T \cdot \mathbf{n} \in [\mathcal{B}'(\partial\Omega)]^N$  by

$$\langle T \cdot \mathbf{n}, \operatorname{Tr} \theta \rangle_{[\mathcal{B}'(\partial\Omega)]^N, [\mathcal{B}(\partial\Omega)]^N} = \int_{\Omega} T : \nabla \theta \, dx - \int_{\Omega} (\operatorname{div} T) \cdot \theta \, dx, \quad \theta \in H^1(\Omega)^N$$

## Variational formulation on admissible domains

$\Gamma_{\text{Dir}}$  and  $\Gamma_{\text{Neu}}$  are two Borel subsets of  $\partial\Omega$  such that

$$\partial\Omega = \Gamma_{\text{Dir}} \cup \Gamma_{\text{Neu}} \text{ and } \mu_{\partial\Omega}(\Gamma_{\text{Dir}} \cap \Gamma_{\text{Neu}}) = \mathbf{0}.$$

A vector field  $\mathbf{u} \in V(\Omega, \Gamma_{\text{Dir}})^N$  is a *weak solution* with data  $\mathbf{f} \in L^2(\Omega)^N$  and  $\mathbf{g} \in L^2(\Gamma_{\text{Neu}})^N$  if

$$\int_{\Omega} \sigma(\mathbf{u}) : \mathbf{e}(\theta) dx = \int_{\Omega} \mathbf{f} \cdot \theta dx + \int_{\Gamma_{\text{Neu}}} \mathbf{g} \cdot \text{Tr} \theta d\mu_{\partial\Omega}, \quad \theta \in V(\Omega, \Gamma_{\text{Dir}})^N.$$

### Theorem

For all  $\mathbf{g} \in [B(\partial\Omega)]^N$  and  $\mathbf{f} \in L^2(\Omega)^N$  there is a unique weak solution  $\mathbf{u} \in V(\Omega, \Gamma_{\text{Dir}})^N$  and  $\exists$  a constant  $C > \mathbf{0}$  such that

$$\|\mathbf{u}\|_{H^1(\Omega)^N} \leq C \left( \|\mathbf{f}\|_{L^2(\Omega)^N} + \|\mathbf{g}\|_{[B(\partial\Omega)]^N} \right).$$

# Korn inequality

- DURAN-2004 for  $(\varepsilon, \infty)$ -domains
- CIARLET-1988 for bounded Lipschitz domains

## Proposition

Let  $\Omega \subset \mathbb{R}^n$  be a bounded  $(\varepsilon, \infty)$ -domain. There is a constant  $C_K(\varepsilon, N, \text{diam}(\Omega)) > 0$  depending only on  $\varepsilon$ ,  $N$  and  $\text{diam}(\Omega)$  such that

$$\|\mathbf{u}\|_{H^1(\Omega)^N} \leq C_K(\varepsilon, N, \text{diam}(\Omega)) \left( \|\mathbf{u}\|_{L^2(\Omega)^N} + \|\mathbf{e}(\mathbf{u})\|_{L^2(\Omega)^{N \times N}} \right)$$

for all  $\mathbf{u} \in H^1(\Omega)^N$ .

The norm  $\mathbf{u} \rightarrow \left( \|\mathbf{u}\|_{L^2(\Omega)^N}^2 + \|\mathbf{e}(\mathbf{u})\|_{L^2(\Omega)^{N \times N}}^2 \right)^{\frac{1}{2}}$  is equivalent to  $\|\cdot\|_{H^1(\Omega)^N}$  on  $H^1(\Omega)^N$ .

# Norm equivalence and Poincaré inequality on admissible and $(\varepsilon, \infty)$ -domains

## Lemma

- (i) Let  $\Gamma_{\text{Dir}}$  is a Borel subset of  $\partial\Omega$  and  $\mu_{\partial\Omega}(\Gamma_{\text{Dir}}) > 0$ . Then  $\exists$  a constant  $c_K(\varepsilon, \mathbf{N}, \mu_{\partial\Omega}, \Gamma_{\text{Dir}}) > 0$  such that

$$\forall \mathbf{u} \in V(\Omega, \Gamma_{\text{Dir}})^N \quad c_K(\varepsilon, \mathbf{N}, \mu_{\partial\Omega}, \Gamma_{\text{Dir}}) \|\mathbf{u}\|_{H^1(\Omega)^N} \leq \|\mathbf{e}(\mathbf{u})\|_{L^2(\Omega)^{N \times N}} \leq \|\mathbf{u}\|_{H^1(\Omega)^N}.$$

- (ii)  $\Omega \subset D \subset \mathbb{R}^N$ ,  $\Gamma_{\text{Dir}}$  is a Borel subset of  $\partial D \cap \partial\Omega$  with  $\mu_{\partial D}(\Gamma_{\text{Dir}}) > 0$  and  $\mu_{\partial\Omega}|_{\Gamma_{\text{Dir}}} = \mu_{\partial D}|_{\Gamma_{\text{Dir}}}$ . Moreover, the Poincaré inequality:  
 $\exists c_K(\varepsilon, \mathbf{N}, D, \mu_{\partial D}, \Gamma_{\text{Dir}}) > 0$  depending only on  $\varepsilon, \mathbf{N}, D, \mu_{\partial D}$  and  $\Gamma_{\text{Dir}}$ .

$$\mathbf{u} \in V(\Omega, \Gamma_{\text{Dir}})^N \quad \int_{\Omega} \mathbf{u} \cdot \mathbf{u} \, dx \leq C_P(\varepsilon, \mathbf{N}, D, \mu_{\partial D}, \Gamma_{\text{Dir}}) \int_{\Omega} \nabla \mathbf{u} : \nabla \mathbf{u} \, dx$$

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Minimum for non-Lipschitz case

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## Mosco convergence, U.Mosco, 1969, 1994

### Definition

A sequence of functionals  $G^m : H \rightarrow (-\infty, +\infty]$  is said to  $M$ -converge to a functional  $G : H \rightarrow (-\infty, +\infty]$  in a Hilbert space  $H$ , if

1. (lim sup condition) For every  $u \in H$  there exists  $u_m$  converging strongly in  $H$  such that

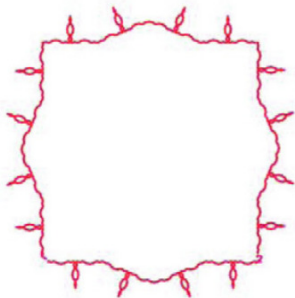
$$\overline{\lim} G^m[u_m] \leq G[u], \quad \text{as } m \rightarrow +\infty. \quad (4)$$

2. (lim inf condition) For every  $v_m$  converging weakly to  $u$  in  $H$

$$\underline{\lim} G^m[v_m] \geq G[u], \quad \text{as } m \rightarrow +\infty. \quad (5)$$

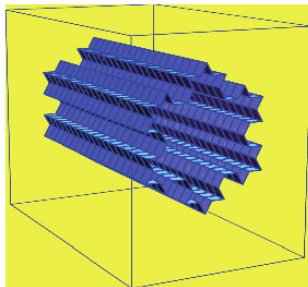
# Approximation of solutions on fractal domains by solutions on prefractal domains (irregular by regular)

- **Von Koch 2D mixtures and snowflake** (mixed Poisson problem, R. Capitanelli, A. Vivaldi, 2010, 2011)



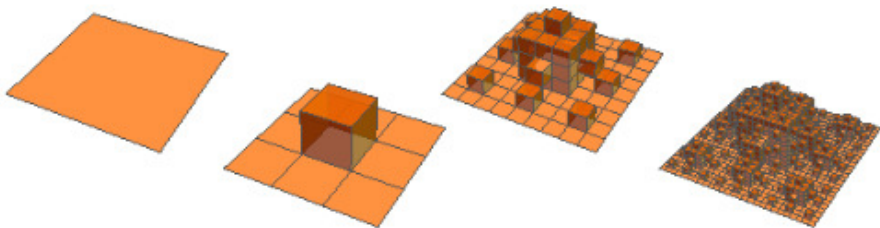
# Approximation of solutions on fractal domains by solutions on prefractal domains (irregular by regular)

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## Approximation of solutions on fractal domains by solutions on prefractal domains (irregular by regular)

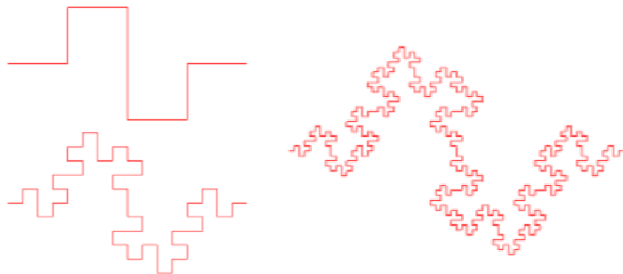
- Von Koch 2D mixtures (mixed Poisson problem, R. Capitanelli, A. Vivaldi, 2010, 2011)
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# Approximation of solutions on fractal domains by solutions on prefractal domains (irregular by regular)

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## Non-monotonous case:



## Mosco convergence result

- Let  $\varepsilon > 0$ , let  $(\Omega_n)_n$  be a sequence of  $(\varepsilon, \infty)$ -domains  $\Omega_n \subset D$ .
- Let all  $\mu_{\Gamma_n}$  satisfy  $d$ -upper regular with the same exponent  $N - 1 < d \leq N$  and the same constant  $c_d$  (as  $\mu_{\Gamma_{\text{Dir}}}$ ).

We define  $\forall \alpha \geq 0$  and  $\alpha_n \geq 0$  quadratic forms on  $L^2(D)^N$  by

$$a_n(\mathbf{u}, \mathbf{u}) = \begin{cases} \int_{\Omega_n} \mathbf{e}(\mathbf{u}) : \mathbf{e}(\mathbf{u}) dx + \int_{\Omega_n} |\mathbf{u}|^2 dx + \alpha_n \int_{\Gamma_n} |\text{Tr} \mathbf{u}|^2 d\mu_{\Gamma_n}, & \mathbf{u} \in V(\Omega_n, \Gamma_{\text{Dir}})^N, \\ +\infty, & \text{otherwise} \end{cases}$$

and

$$a(\mathbf{u}, \mathbf{u}) = \begin{cases} \int_{\Omega} \mathbf{e}(\mathbf{u}) : \mathbf{e}(\mathbf{u}) dx + \int_{\Omega} |\mathbf{u}|^2 dx + \alpha \int_{\Gamma} |\text{Tr} \mathbf{u}|^2 d\mu_{\Gamma}, & \mathbf{u} \in V(\Omega, \Gamma_{\text{Dir}})^N, \\ +\infty, & \text{otherwise.} \end{cases}$$

## Mosco convergence result

### Theorem

If  $\lim_n \Omega_n = \Omega$  in the Hausdorff sense:

$$m \in \mathbb{N} \quad \Omega_m \subset D \quad \text{and} \quad d_H(D \setminus \Omega_m, D \setminus \Omega) \rightarrow 0 \quad \text{for} \quad m \rightarrow +\infty$$

and in the sense of characteristic functions:

$$\mathbf{1}_{\Omega_m} \rightarrow \mathbf{1}_{\Omega} \quad \text{for} \quad m \rightarrow +\infty \quad \text{in} \quad L^p_{loc}(\mathbb{R}^n) \quad \forall p \in [1, \infty[$$

and  $\lim_n \alpha_n \mu_n = \alpha \mu$  in the sense of weak convergence:

$$\forall \phi \in C(\bar{D}) \quad \int_{\Gamma_m} \alpha_n \phi d\mu_n \rightarrow \int_{\Gamma} \alpha \phi d\mu, \quad n \rightarrow +\infty$$

then we have

$$\lim_n \mathbf{a}_n = \mathbf{a} \quad \text{in the sense of } M\text{-convergence on } L^2(D).$$

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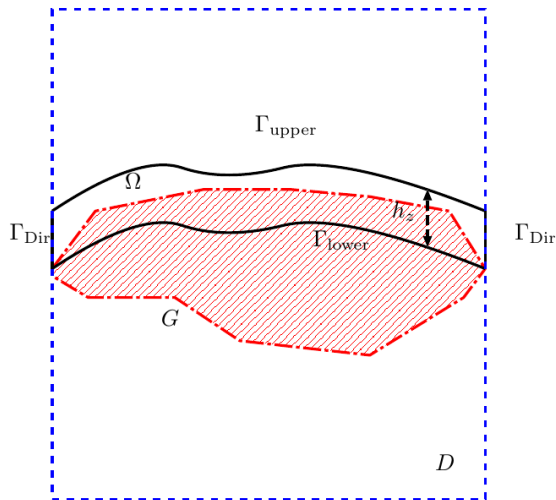
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## Open problems and perspectives

# Optimization problem: loads of snow on roofs



# Optimization framework



$$\Omega \mapsto J(\Omega, \mu_{\partial\Omega}, \mathbf{v}) := c_1 \int_{\Omega} |\mathbf{v}|^2 dx + c_2 \int_{\Omega} \mathbf{Ae}(\mathbf{v}) : \mathbf{e}(\mathbf{v}) dx, \quad \mathbf{v} \in V(\Omega, \Gamma_{\text{Dir}})^N, \quad c_1 \geq 0, \quad c_2 \geq 0$$

## The most stable (strongest) roof

- Roof is  $\Omega \subset \mathbb{R}^3$ .
- The moving part  $\Gamma_{\text{Neu}}$  of the boundary  $\partial\Omega$  is the union of two disjoint parts  $\Gamma_{\text{upper}}$  and  $\Gamma_{\text{lower}}$  which model the upper and the lower side of the roof.
- The set  $\Gamma_{\text{Dir}}$  models vertical boundary parts of the roof at which it is fixed by a structural connection to other parts of the building.
- The volume  $c_v > 0$  of the roof is fixed;
- $\Gamma_{\text{upper}}$  and  $\Gamma_{\text{lower}}$  are always parallel with a vertical distance

$$h_z(\Omega) = \frac{c_v}{\mu(\Gamma_{\text{lower}})}. \quad \mathbf{f} = \rho_0 h_z(\Omega) \mathbf{e}_z,$$

where  $\rho_0 > 0$  is a constant (mass) density and  $\mathbf{e}_z = (0, 0, 1)$ .

- The field  $\mathbf{f}(\Omega)$  represents the weight of the roof  $\Omega$ .

## Admissible class of domains $U_{ad}$

**CHENAIS-1975** Domains in a fixed ball satisfying for  $\varepsilon > 0$  the  $\varepsilon$ -cone property:

$\forall x \in \partial\Omega, \exists \xi_x \in \mathbb{R}^n$  with  $\|\xi_x\| = 1$  such that for all  $y \in \bar{\Omega} \cap B_\varepsilon(x)$

$C(y, \xi_x, \varepsilon) = \{z \in \mathbb{R}^n \mid (z-y, \xi_x) \geq \cos(\varepsilon)\|z-y\| \text{ and } 0 < \|z-y\| < \varepsilon\} \subset \Omega.$

## Admissible class of domains $U_{ad}$

**CHENAIS-1975** Domains in a fixed ball satisfying for  $\varepsilon > 0$  the  $\varepsilon$ -cone property:

**Compact class** for

1. the Hausdorff convergence of domains:

$$m \in \mathbb{N} \quad \Omega_m \subset D \quad \text{and} \quad d_H(D \setminus \Omega_m, D \setminus \Omega) \rightarrow 0 \quad \text{for} \quad m \rightarrow +\infty$$

2. in the sense of characteristic functions:

$$\mathbf{1}_{\Omega_m} \rightarrow \mathbf{1}_{\Omega} \quad \text{for} \quad m \rightarrow +\infty \quad \text{in} \quad L^p_{loc}(\mathbb{R}^n) \quad \forall p \in [1, \infty[$$

3. in the sense of compacts:

$$\begin{cases} \forall K \text{ compact in } \Omega & \Rightarrow & K \subset \Omega_m \\ \forall U \text{ compact in } D \setminus \bar{\Omega} & \Rightarrow & U \subset D \setminus \bar{\Omega}_m \end{cases} \quad \text{for a sufficiently large } m$$

## Admissible class of domains $U_{ad}$

**CHENAIS-1975** Domains in a fixed ball satisfying for  $\varepsilon > 0$  the  $\varepsilon$ -cone property:  
**Compact class** for three type of domain convergences.

**But we don't have**

$$\forall f \in C(\bar{D}) \quad \int_{\partial\Omega_m} f d\mathcal{H}^{n-1} \rightarrow \int_{\partial\Omega} f d\mathcal{H}^{n-1}.$$

We have

- the continuity of the volume
- **lower semi-continuity of perimeters**

**Existence of optimal shapes for homogeneous Dirichlet problems.**

## Admissible class of domains $U_{ad}$

**CHENAIS-1975** Domains in a fixed ball satisfying for  $\varepsilon > 0$  the  $\varepsilon$ -cone property:

- **Compact class** for three type of domain convergences
- $E_m : H^1(\Omega_m) \rightarrow H^1(\mathbb{R}^n)$  have a norm uniformly bounded (independent on  $m$ ).

**Existence of optimal shapes for homogeneous Neumann problems.**

## Admissible class of domains $U_{ad}$

**CHENAIS-1975** Domains in a fixed ball satisfying for  $\varepsilon > 0$  the  $\varepsilon$ -cone property

**BUCUR-2016** Relaxation method for Lipschitz boundaries of finite  $\mathcal{H}^{n-1}$  measure  
(free boundary discontinuous problems)

Robin boundary problems

**No-compactness  $\implies$  no existence result.**

## Lipschitz admissible domains: $\mu := \mathcal{H}^{n-1}$ (for $\Omega \subset \mathbb{R}^n$ )

### Definition

1. Let  $\mathcal{O}(D, \varepsilon)$  be the class of all domains  $\Omega \subset D$  satisfying for the same  $\varepsilon > 0$  the  $\varepsilon$ -cone property [AGMON-1965, CHENAIS-1975]
2.  $G$  is a nonempty open proper subset of  $D$ ,  $\Gamma_{\text{Dir}}$  a compact subset of  $\partial D$  with  $\mathcal{H}^{N-1}(\Gamma_{\text{Dir}}) > 0$  and  $\varepsilon > 0$ ,  $c_V > 0$ ,  $\hat{c} > 0$  are given constants.

$$U_{\text{ad}}(D, G, \varepsilon, c_V, \hat{c}, \Gamma_{\text{Dir}}) = \{\Omega \in \mathcal{O}(D, \varepsilon) \mid \lambda^N(\Omega) = c_V,$$

$$\partial\Omega = \Gamma_{\text{Dir}} \cup \Gamma_{\text{Neu}}, \mathcal{H}^{N-1}(\Gamma_{\text{Dir}} \cap \Gamma_{\text{Neu}}) = 0, \Gamma_{\text{Dir}} = \partial\Omega \cap \partial D, \Gamma_{\text{Neu}} \subset \bar{G}$$

$$\text{and } \boxed{\mathcal{H}^{N-1}(\Gamma_{\text{Neu}} \cap B(x, r)) \leq \hat{c}r^{N-1}} \text{ for any } x \in \Gamma_{\text{Neu}} \text{ and } r > 0\}.$$

# Existence of optimal shape

## Theorem

Let  $\mathbf{c}_1 \geq \mathbf{0}$ ,  $\mathbf{c}_2 \geq \mathbf{0}$ ,  $\mathbf{f} \in L_2(D)^N$  and  $\mathbf{g} \in [B(\partial D)]^N$  be fixed. Then

1.  $\mathbf{U}_{ad}(D, \mathbf{G}, \varepsilon, \mathbf{c}_V, \hat{\mathbf{c}}, \Gamma_{Dir})$  is compact with respect to the Hausdorff convergence, in the sense of characteristic functions in  $L^1(D)$  and in the sense of compacts.
2. There are a domain  $\Omega_{opt} \in \mathbf{U}_{ad}(D, \mathbf{G}, \varepsilon, \mathbf{c}_V, \hat{\mathbf{c}}, \Gamma_{Dir})$  and a finite Borel measure  $\mu_{\partial\Omega_{opt}}$  on  $\partial\Omega_{opt}$ , equivalent to  $\mathcal{H}^{N-1}|_{\partial\Omega_{opt}}$ , such that  $\mathcal{H}^{N-1}(\Gamma_{Neu,opt}) \leq \mu_{\partial\Omega_{opt}}(\Gamma_{Neu,opt})$  and

$$\begin{aligned}
 J(\Omega_{opt}, \mu_{\partial\Omega_{opt}}, \mathbf{u}(\Omega_{opt}, \mu_{\partial\Omega_{opt}})) \\
 = \inf_{\Omega \in \mathbf{U}_{ad}(D, \mathbf{G}, \varepsilon, \mathbf{c}_V, \hat{\mathbf{c}}, \Gamma_{Dir})} J(\Omega, \mathcal{H}^{N-1}, \mathbf{u}(\Omega, \mathcal{H}^{N-1})).
 \end{aligned}$$

If  $\mu_{\partial\Omega_{opt}} = \mathcal{H}^{N-1}|_{\partial\Omega_{opt}}$ , then the infimum is actually a minimum.

## General uniform admissible domains: $U_{ad}(D, D_0, \varepsilon, s, d, \bar{c}_s, c_d)$

### Definition $((\varepsilon, \infty)$ -domain or uniform domain)

$\forall x, y \in \Omega$  there is a rectifiable arc  $\gamma \subset \Omega$  with length  $\ell(\gamma)$

joining  $x$  to  $y$  and satisfying

1.  $\ell(\gamma) \leq \frac{|x-y|}{\varepsilon}$  (**uniformly locally quasiconvex**) and
2.  $d(z, \partial\Omega) \geq \varepsilon|x-z| \frac{|y-z|}{|x-y|}$  for  $z \in \gamma$ .

**Examples:** Lipschitz domains, von Koch snow-flake, ...

no-collapsing domains, NTA domains

**Uniform domains are Sobolev extension domains by JONES-1981.**

## General uniform admissible domains: $U_{ad}(D, D_0, \varepsilon, s, d, \bar{c}_s, c_d)$

### Definition

Let  $D_0 \subset D \subset \mathbb{R}^n$  be non-empty bounded Lipschitz domains.

A pair  $(\Omega, \mu)$  is called a **shape admissible domain** with parameters

$$D, \quad D_0, \quad \varepsilon > 0, \quad n-1 \leq s < n, \quad 0 \leq d \leq s, \quad \bar{c}_s > 0, \quad c_d > 0$$

if

1.  $\Omega$  is an  $(\varepsilon, \infty)$ -domain:  $D_0 \subset \Omega \subset D$
2.  $\mu$  is a finite Borel measure  $\mu$ ,  $\text{supp} \mu = \partial\Omega$ :

$$\mu(B_r(x)) \leq c_d r^d, \quad x \in \partial\Omega, \quad 0 < r \leq 1, \quad (\Rightarrow \dim_H \partial\Omega \geq d)$$

$$\mu(\overline{B_r(x)}) \geq \bar{c}_s r^s, \quad x \in \partial\Omega, \quad 0 < r \leq 1.$$

(If  $s = d$  then  $\dim_H \partial\Omega = d$  and  $\partial\Omega$  is  $d$ -set).

# Existence of optimal shape in the general case

## Theorem

1.  $U_{ad}(D, D_0, \varepsilon, \mathbf{s}, \mathbf{d}, \bar{\mathbf{c}}_s, \mathbf{c}_d)$  is compact with respect to the Hausdorff convergence, in the sense of characteristic functions in  $L^1(D)$ , in the sense of compacts and **weak convergence of boundary measures**.
2. Let  $\mathbf{c}_1 \geq \mathbf{0}$ ,  $\mathbf{c}_2 \geq \mathbf{0}$ ,  $\mathbf{f} \in L_2(D)^N$  and  $\mathbf{g} \in [B(\partial D)]^N$  be fixed. Then  $\exists (\Omega_{opt}, \mu^*) \in U_{ad}$ :

$$\min_{\Omega \in U_{ad}(D, D_0, \varepsilon, \mathbf{s}, \mathbf{d}, \bar{\mathbf{c}}_s, \mathbf{c}_d)} J(\Omega, \mathbf{u}(\Omega, \mu), \mu) = J(\Omega_{opt}, \mathbf{u}(\Omega_{opt}, \mu^*), \mu^*)$$

for  $\mathbf{u}$ , the weak solution of the linear elasticity problem.

## Existence of optimal shape for the roofs

Let  $N = 3$  and let  $D \subset \mathbb{R}^3$  be a bounded Lipschitz domain,  $G$  a nonempty open proper subset of  $D$  and  $\Gamma_{\text{Dir}}$  a compact subset of  $\partial D$  with  $\mathcal{H}^2(\Gamma_{\text{Dir}}) > 0$ . Consider the class

$$\begin{aligned}
 U_{\text{ad}}(D, G, \varepsilon, \mathbf{c}_V, \mathbf{s}, \mathbf{d}, \bar{\mathbf{c}}_S, \mathbf{c}_d, \Gamma_{\text{Dir}}) &= \{(\Omega, \mu_{\partial\Omega}) \mid \Omega \in \hat{\mathcal{O}}(D, \varepsilon), \lambda^3(\Omega) = \mathbf{c}_V, \\
 &\quad \partial\Omega = \Gamma_{\text{Dir}} \cup \Gamma_{\text{Neu}}, \Gamma_{\text{Dir}} = \partial\Omega \cap \partial D, \mu_{\partial\Omega} = \mathcal{H}^{N-1}|_{\Gamma_{\text{Dir}}} + \mu_{\Gamma_{\text{Neu}}}, \\
 &\quad \mu_{\Gamma_{\text{Neu}}} \text{ with } \text{supp } \mu_{\Gamma_{\text{Neu}}} = \Gamma_{\text{Neu}} \text{ satisfies } \mathbf{d}\text{-upper and } \mathbf{s}\text{-lower regular conditions,} \\
 &\quad \mu_{\partial\Omega}(\Gamma_{\text{Dir}} \cap \Gamma_{\text{Neu}}) = 0, \\
 &\quad \Gamma_{\text{Neu}} = \Gamma_{\text{lower}} \cup \Gamma_{\text{upper}}, \Gamma_{\text{lower}} \subset \bar{G}, \Gamma_{\text{upper}} = \Gamma_{\text{lower}} + h_{\mathbf{z}}\mathbf{e}_{\mathbf{z}}\}.
 \end{aligned}$$

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## Open problems and perspectives

## Open problems and perspectives

We don't have  $H^2$ -regularity of weak solutions of the elliptic problems

1. Robin type conditions
2. Models described by  $\Delta^2$
3. numerical algorithms adapted for approximations of fractal boundary measures
4. Shape derivatives over fractal/non-Lipschitz boundaries

**Rough boundaries are the energy minimizers.**

Thank you very much for your attention!