The De Giorgi variational principle for gradient flows: a direct approach

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Plan of the talk

1. **Variational principles for evolution equations**

2. **De Giorgi's principle and main result**

3. **Three key arguments**
   - Relaxation and continuity equation
   - Von Neumann duality
   - Backward estimates and maximum principle for first order HJ equations

4. **The infinite-dimensional setting**

5. **Extensions and open problems**

Outline

Gradient flows and doubly nonlinear equations: a simple smooth model

Let $X = \mathbb{R}^d$, $\varphi\colon \mathbb{R}^d \to \mathbb{R}$ be a smooth energy function, $x_0 \in \mathbb{R}^d$ be an initial datum.

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More generally:

Doubly nonlinear equation

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where Ψ is a smooth convex *dissipation potential*, superlinear, $\Psi(0) = 0$.

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Simple finite dimensional, smooth setting: **existence by Cauchy-Lipschitz Theorem.**

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Key question: how to prove existence of solutions via *global-in-time variational methods*?

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Fenchel inequality: $\Phi(x) + \Phi^*(v) \geqslant \langle v, x\rangle$,

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Examples:

- $\Phi(x) = \frac{1}{2}|x|^2 \Rightarrow \Phi^*(v) = \frac{1}{2}|v|^2$.
- $\Phi(x) = \frac{1}{p}\|x\|^p \Rightarrow \Phi^*(v) = \frac{1}{q}\|v\|_*^q, \quad \frac{1}{p} + \frac{1}{q} = 1$.
- $\Phi(x) = x \log x - x \Rightarrow \Phi^*(v) = e^v$.

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Rewriting the gradient flow: $\dot{x} = -D\varphi(x) \iff -\dot{x} = D\varphi(x) \iff x = D\varphi^*(-\dot{x})$, i.e.

$$-D\varphi^*(-\dot{x}(t)) = -x(t) \quad \iff \quad D\Psi(\dot{x}(t)) = -D\Phi(x(t))$$

with $\Psi(v) := \varphi^*(-v)$ and $\Phi(x) := \frac{1}{2}|x|^2$.

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$$\mathcal{BE}(T, x) := \frac{1}{2}|x(T)|^2 + \int_0^T \Big(\varphi^*(-\dot{x}(\tau)) + \varphi(x(\tau))\Big)\, d\tau - \frac{1}{2}|x_0|^2.$$

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Observation: Fenchel's inequality gives

$$\varphi(x(t)) + \varphi^*(-\dot{x}(t)) \geqslant -\langle x(t), \dot{x}(t)\rangle = -\frac{d}{dt}\frac{1}{2}|x(t)|^2,$$

with **equality** $\iff$ $-\dot{x}(t) = D\varphi(x(t))$ (i.e. x is a gradient flow).

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Hence: $\mathcal{BE}(T, x) \geqslant 0$, and $\mathcal{BE}(T, x) = 0 \iff x$ solves the gradient flow.

Null-minimization: solutions = null-minimizers of $\mathcal{BE}$.

Proving $\min \mathcal{BE} = 0$: Ghoussoub–Tzou/McCann

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Via *self-dual Lagrangians* and duality, they prove $\min \mathcal{BE} = 0$ for:

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Key features of Brezis–Ekeland:

- The space is **Hilbert**.
- The "Lyapunov functional" is always $\frac{1}{2}|x|^2$ (the *energy* φ enters through $\varphi + \varphi^*$).
- The self-dual structure requires **convexity** of φ.

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A different variational characterization, where the Lyapunov functional is the **energy** φ itself.

For gradient flows ($\dot{x} = -\mathrm{D}\varphi(x)$):

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$$\mathcal{J}(T, x) := \varphi(x(T)) + \int_0^T \left(\frac{1}{2}|\dot{x}|^2 + \frac{1}{2}\left| -\mathrm{D}\varphi(x) \right|^2 \right) \mathrm{d}\tau - \varphi(x_0).$$

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Why $\mathcal{J} \geqslant 0$? By Young's inequality $\frac{1}{2}|a|^2 + \frac{1}{2}|b|^2 \geqslant \langle a, b\rangle$:

$$\tfrac{1}{2}\big|\dot{x}\big|^2 + \tfrac{1}{2}\big| - D\varphi(x)\big|^2 \geqslant \langle - D\varphi(x), \dot{x}\rangle = -\frac{d}{dt}\varphi(x(t)),$$

with **equality** $\iff$ $\dot{x} = -D\varphi(x)$. Integrating: $\mathcal{J} \geqslant 0$, and $\mathcal{J} = 0$ $\iff$ x solves the gradient flow.

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Null-minimization: solutions = null-minimizers of $\mathcal{J}$.

The hard part: prove $\min \mathcal{J} \leqslant 0$ without knowing a solution exists!

The De Giorgi functional for doubly nonlinear equations

For $D\Psi(\dot{x}) = -D\varphi(x)$, the same construction with **Fenchel inequality** replacing Young:

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Indeed: $\Psi(v) + \Psi^*(-z) \geqslant \langle -z, v\rangle$ gives

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The **slope** $S(x) := \Psi^*(-D\varphi(x))$ (for GF: $\frac{1}{2}|a|^2 + \frac{1}{2}|b|^2 \rightsquigarrow S = \frac{1}{2}|D\varphi|^2$).

Proposition (De Giorgi's principle)

(a) $\mathcal{J}(T, \boldsymbol{x}) \geqslant 0$ *for every* $\boldsymbol{x}$ *with* $\boldsymbol{x}(0) = x_0$.

(b) $\mathcal{J}(T, \boldsymbol{x}) = 0 \iff \boldsymbol{x}$ *solves* $D\Psi(\dot{\boldsymbol{x}}) = -D\varphi(\boldsymbol{x})$.

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Fundamental differences with Brezis–Ekeland:

- **No convexity** of φ required.
- The Lyapunov functional is **the energy** φ, not $\frac{1}{2}|x|^2$.
- Works in **Banach** and **metric** spaces.
- However $\mathcal{J}$ is **not convex** in general, even for convex φ, due to the presence of $\Psi^*(-D\varphi)$.

The Minimizing Movement / JKO scheme

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$$x_k^\tau \in \arg\min_x \Big\{ \tau \Psi\Big(\frac{x - x_{k-1}^\tau}{\tau}\Big) + \varphi(x) \Big\}, \quad k = 1, 2, \ldots \qquad \rightsquigarrow \quad D\Psi\Big(\frac{x - x_{k-1}^\tau}{\tau}\Big) + D\varphi(x) = 0.$$

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For gradient flows ($\Psi = \frac{1}{2}|\cdot|^2$): $\quad x_k^\tau \in \arg\min_x \left\{ \frac{|x - x_{k-1}^\tau|^2}{2\tau} + \varphi(x) \right\}.$

This is the **JKO/Minimizing Movement scheme** (Jordan–Kinderlehrer–Otto, 1998; De Giorgi, 1993).

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Our approach: prove $\min \mathcal{J} \leqslant 0$ *directly*, without time discretization, via convexification + duality.

The main theorem (smooth Euclidean setting)

Additional assumptions: $\varphi\colon \mathbb{R}^d \to [0,+\infty)$ smooth, coercive: $\varphi(x) \to +\infty$, $|D\varphi(x)| \to +\infty$ as $|x| \to +\infty$. **No convexity!** $\Rightarrow S(x) = \Psi^*(-D\varphi(x))$ coercive.

Variational argument (Pinzi–Riva–S.)

For all $x_0 \in \mathbb{R}^d$:

$$\boxed{\min\left\{\mathcal{J}(T,\boldsymbol{x}) : \boldsymbol{x} \in AC([0,T];\mathbb{R}^d),\ \boldsymbol{x}(0) = x_0\right\} = 0.}$$

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Three ingredients:

1. Relaxation via continuity equation and superposition principle.
2. Von Neumann minimax and the dual problem.
3. Backward (upper) estimates for solutions to first order Hamilton-Jacobi equations.

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Natural convexification: consider probability measures $\lambda \in \mathscr{P}(AC([0,T];\mathbb{R}^d))$ on curves with $\boldsymbol{x}(0) = x_0$, and define

$$\bar{\mathcal{J}}(\lambda) := \int_{AC} \mathcal{J}(T, \boldsymbol{x})\, d\lambda(\boldsymbol{x}).$$

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Since $\mathcal{J}(\mathbf{x}) = \overline{\mathcal{J}}(\delta_{\mathbf{x}})$: $\quad \min_\lambda \overline{\mathcal{J}} \leqslant \min_{\mathbf{x}} \mathcal{J}$.

Conversely, for any λ: $\overline{\mathcal{J}}(\lambda) = \int \mathcal{J}\, d\lambda \geqslant \min \mathcal{J}$.

Therefore: $\min_\lambda \overline{\mathcal{J}} = \min_{\mathbf{x}} \mathcal{J}$. **Convexification does not change the minimum.**

From measures on curves to curves of measures

A curve x with $x(0) = x_0$ induces a triplet of measures on $X_T = [0,T] \times \mathbb{R}^d$:

$$m_x = \delta_{x(T)}, \quad \mu_x = \mathcal{L}^1 \otimes \delta_{x(t)}, \quad \nu_x = \dot{x}(t)\, \mathcal{L}^1 \otimes \delta_{x(t)} = \boldsymbol{v} \cdot \mu_x$$

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$$m_x = \delta_{x(T)}, \quad \mu_x = \mathcal{L}^1 \otimes \delta_{x(t)}, \quad \nu_x = \dot{x}(t)\, \mathcal{L}^1 \otimes \delta_{x(t)} = \boldsymbol{v} \cdot \mu_x$$

They satisfy the **continuity equation** in the sense of distributions:

$$\partial_t \mu_x + \operatorname{div} \nu_x = 0 \quad \text{in } (0, T) \times \mathbb{R}^d, \quad \mu_x(0, \cdot) = \delta_{x_0}, \ \mu_x(T, \cdot) = m_x;$$

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Given $\lambda \in \mathscr{P}(AC)$, integrate: $\quad m_\lambda = \int m_x \, d\lambda, \ \ \mu_\lambda = \int \mu_x \, d\lambda, \ \ \nu_\lambda = \int \nu_x \, d\lambda.$

$(m_\lambda, \mu_\lambda, \nu_\lambda)$ still satisfies the CE (linearity); $\nu = \boldsymbol{v}\mu$.

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The link between position and velocity is **transferred to the continuity equation**.

Relaxed functional: over triplets (m, μ, ν), $\nu = \boldsymbol{v}\mu$, satisfying the CE,

$$\mathcal{E}(m, \mu, \nu) := \int_{\mathbb{R}^d} \varphi \, dm + \int_{X_T} \big(\Psi(\boldsymbol{v}(t, x)) + \Psi^*(-D\varphi(x))\big)\, d\mu - \varphi(x_0); \qquad \boldsymbol{v} = \frac{d\nu}{d\mu}$$

By Jensen: $\mathcal{E}(m_\lambda, \mu_\lambda, \nu_\lambda) \leqslant \overline{\mathcal{J}}(\lambda)$. **But could we lose something?**

Superposition principle (Smirnov; Ambrosio; Ambrosio–Gigli–S.):

Any solution (m, μ, ν), with $\nu = \boldsymbol{v}\mu$, of the continuity equation can be *decomposed* into absolutely continuous curves: there exists $\lambda \in \mathscr{P}(AC([0,T];\mathbb{R}^d))$ such that

$$m = m_\lambda, \quad \mu = \mu_\lambda, \quad \nu = \nu_\lambda,$$

and the curves satisfy $\dot{x}(t) = \boldsymbol{v}(t, \boldsymbol{x}(t)) \quad \lambda\text{-a.e.}$

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Therefore: $\mathcal{E}(m, \mu, \nu) = \displaystyle\int_{AC} \mathcal{J}(T, \boldsymbol{x})\, d\lambda(\boldsymbol{x}) = \overline{\mathcal{J}}(\lambda) \geqslant \min \overline{\mathcal{J}}$.

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Putting everything together:

$$\boxed{\min_{\boldsymbol{x}} \mathcal{J} = \min_{\lambda} \overline{\mathcal{J}} = \min_{(m,\mu,\nu)\in CE} \mathcal{E}.}$$

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No loss! It suffices to show $\min \mathcal{E} \leqslant 0$ for the **convex** functional $\mathcal{E}$.

The saddle-point formulation

Encode the continuity equation constraint via a **Lagrange multiplier** $\xi \in C^1_b(X_T)$:

$$\mathcal{C}(\xi, m, \mu, \nu) := \int_{X_T} \partial_t \xi \, d\mu + \int_{X_T} D_x \xi \cdot d\nu - \int_{\mathbb{R}^d} \xi(T) \, dm + \xi(0, x_0).$$

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Lagrangian: $\mathcal{L}(\xi, m, \mu, \nu) = \mathcal{E}(m, \mu, \nu) + \mathcal{C}(\xi, m, \mu, \nu)$.

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Here: $\mathcal{L}$ is **convex** in (m, μ, ν) and **linear** (hence concave) in ξ. ✓

Verification of the minimax hypotheses

Theorem (Von Neumann minimax)

Let $\mathcal{L}(\xi, \theta)$ *with* $\xi \in \mathrm{A}$, $\theta \in \mathrm{B}$ *convex sets,* B *endowed with a Hausdorff topology. Assume:*

- $\xi \mapsto \mathcal{L}(\xi, \theta)$ *is* **concave** *for every* θ*;*
- $\theta \mapsto \mathcal{L}(\xi, \theta)$ *is* **convex** *and* **l.s.c.** *for every* ξ*;*
- **coercivity:** *there exist* $\overline{\mathrm{C}} > \sup_{\mathrm{A}} \inf_{\mathrm{B}} \mathcal{L}$ *and* $\bar{\xi} \in \mathrm{A}$ *such that* $\{\mathcal{L}(\bar{\xi}, \cdot) \leqslant \overline{\mathrm{C}}\}$ *is nonempty and compact.*

Then: $\displaystyle \min_{\theta \in \mathrm{B}} \sup_{\xi \in \mathrm{A}} \mathcal{L}(\xi, \theta) = \sup_{\xi \in \mathrm{A}} \inf_{\theta \in \mathrm{B}} \mathcal{L}(\xi, \theta).$

No topology on A needed!

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Then: $$\min_{\theta \in B} \sup_{\xi \in A} \mathcal{L}(\xi, \theta) = \sup_{\xi \in A} \inf_{\theta \in B} \mathcal{L}(\xi, \theta).$$

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In our setting: $\xi \in A = C^1_b(X_T)$, $\quad \theta = (m, \mu, \nu) \in B$ with narrow topology.

- Concavity in ξ: $\mathcal{L}$ is linear in ξ. ✓
- Convexity + l.s.c. in θ: convexity of Ψ. ✓
- **Compactness of sublevels** (Prokhorov): coercivity of φ and $S \Rightarrow$ tightness of m, μ; superlinearity of $\Psi \Rightarrow$ tightness of ν. ✓

The dual problem

After switching inf and sup and computing:

$$\inf_{(m,\mu,\nu)} \mathcal{L}(\xi, m, \mu, \nu) = \begin{cases} \xi(0, x_0) - \varphi(x_0), & \text{if } \xi \in \mathsf{HJ}, \\ -\infty, & \text{otherwise,} \end{cases}$$

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where $\xi \in \mathrm{HJ}$ means that ξ satisfies the **Hamilton–Jacobi backward differential inequality**:

$$\boxed{\begin{cases} -\partial_t \xi(t,x) + \Psi^*(-D_x \xi(t,x)) \leqslant \Psi^*(-D\varphi(x)), \\ \xi(T,x) \leqslant \varphi(x). \end{cases}}$$

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Hence the problem reduces to:

$$\min \mathcal{J} = \sup \Big\{ \xi(0,x_0) - \varphi(x_0) : \xi \in \mathrm{HJ} \Big\}.$$

It suffices to show the **r.h.s.** $\leqslant 0$.

[Links with optimal transport, first order mean field games, mean field planning, (mean field) optimal control: Otto-Villani, Cardaliaguet, Santambrogio, Porretta, ...]

The backward estimate (comparison/maximum principle for HJ)

Backward (upper) estimate

Any $\xi \in \mathrm{HJ}$ satisfies $\xi(t,x) \leqslant \varphi(x)$ for *all* $(t,x) \in X_T$.

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Interpretation (maximum principle): φ is a trivial, constant in time, solution of the associated Hamilton–Jacobi *equation*.

$$-\partial_t \varphi(x) + \Psi^*(-D_x\varphi(x)) = \Psi^*(-D\varphi(x)),$$

If the subsolution ξ satisfies $\xi(T,\cdot) \leqslant \varphi$ at the final time, the HJ dynamics propagates this bound *backward to all earlier times*.

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Consequence: $\xi(0,x_0) \leqslant \varphi(x_0)$, hence

$$\sup\left\{\xi(0,x_0) - \varphi(x_0) : \xi \in \mathrm{HJ}\right\} \leqslant 0 \quad \Longrightarrow \quad \boxed{\min \mathcal{J} \leqslant 0.}$$

Proof of the backward estimate

By contradiction. Perturbation: for $\varepsilon > 0$, set $\xi_\varepsilon(t, x) := \xi(t, x) + \varepsilon(t - T - 1)$ (strict HJ inequality).

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By coercivity of φ, $\bar{t}$ is attained at some $\bar{x}$. Then:

- $\bar{t} < T$ (from the terminal condition $\xi_\varepsilon(T,x) \leqslant \varphi(x) - \varepsilon$);
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Plugging into the HJ inequality:

$$0 \leqslant -\partial_t \xi_\varepsilon(\bar{t},\bar{x}) = -\partial_t \xi_\varepsilon(\bar{t},\bar{x}) + \underbrace{\Psi^*(-D_x\xi_\varepsilon(\bar{t},\bar{x})) - \Psi^*(-D\varphi(\bar{x}))}_{=0}$$

$< 0.$ Contradiction!

Dynamic programming viewpoint

The connection with HJ is natural from **optimal control**.

The **value function**

$$V(t,x) := \inf \left\{ \int_t^T \Big(\Psi(v) + \Psi^*(-D\varphi(\mathbf{x})) \Big) d\tau + \varphi(\mathbf{x}(T)) : \dot{\mathbf{x}} = v,\ \mathbf{x}(t) = x \right\}$$

solves the Hamilton–Jacobi equation

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But: this requires smoothness and uniqueness of HJ.

Our approach replaces *uniqueness* with a *comparison principle* for **subsolutions** — works in nonsmooth, infinite-dimensional settings.

Outline

1. Variational principles for evolution equations
2. De Giorgi's principle and main result
3. Three key arguments
   - Relaxation and continuity equation
   - Von Neumann duality
   - Backward estimates and maximum principle for first order HJ equations
4. **The infinite-dimensional setting**
5. Extensions and open problems

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So far: $\mathbb{R}^d$, smooth φ and Ψ.

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Even the Hilbert gradient flow $\dot{x} + \partial\varphi(x) \ni 0$

is nontrivial: φ is **not assumed convex** nor a quadratic perturbation of a convex function.

(Colli–Visintin; Colli; DiBenedetto–Showalter; Rossi–S.; Mielke–Rossi–S.)

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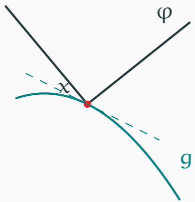

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In the proof, the test function ξ touches φ from below at the critical point, giving $D_x\xi \in \partial\varphi(\bar{x})$.

X reflexive separable Banach space, $\varphi\colon X \to (-\infty, +\infty]$ l.s.c.

Fréchet (viscosity) subdifferential

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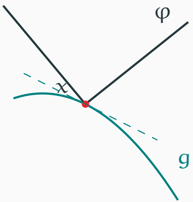

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Assumption: $\partial\varphi$ is *strong-weak** *sequentially closed*.

Then $S(x) = \min_{z\in\partial\varphi(x)} \Psi^*(-z)$ is l.s.c.

General assumptions and the slope

$\varphi\colon X \to (-\infty, +\infty]$ l.s.c., with boundedly compact sublevels, bounded below.

$\Psi\colon X \to (-\infty, +\infty]$ convex, l.s.c., superlinear, *finite in a neighborhood of* 0: Ψ^* is **weakly $*$ coercive** (bounded sublevels).

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If S *has* **compact sublevels**, *then*

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Compactness of sublevels of S ensures tightness of μ in the Von Neumann argument (replacing the coercivity of $|D\varphi|$ used in $\mathbb{R}^d$).

The exponentially weighted functional

When S *does not* have compact sublevels, we gain coercivity via an **exponential weight** ($a > 0$).

Trick: differentiate $e^{-at}\varphi(\mathbf{x}(t))$ instead of $\varphi(\mathbf{x}(t))$: $\frac{d}{dt}\big(e^{-at}\varphi(\mathbf{x})\big) = e^{-at}\big(\langle z, \dot{\mathbf{x}}\rangle \boldsymbol{-a\,\varphi(\mathbf{x})}\big)$.

Weighted functional:

$$\mathcal{J}^a(t, \mathbf{x}) := e^{-at}\varphi(\mathbf{x}(t)) + \int_0^t e^{-a\tau}\Big(\Psi(\dot{\mathbf{x}}(\tau)) + S(\mathbf{x}(\tau)) \boldsymbol{+a\,\varphi(\mathbf{x}(\tau))}\Big)\,d\tau - \varphi(x_0).$$

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For $a = 0$: recovers the first version.

For $\boldsymbol{a > 0}$: coercivity of φ is sufficient.

The chain rule and solutions

Chain Rule (CR)

For every $x \in AC([0,T];X)$ with $\int_0^T (\Psi(\dot{x}) + S(x))\,d\tau < +\infty$,

$t \mapsto \varphi(x(t))$ is absolutely continuous and $\frac{d}{dt}\varphi(x(t)) = \langle z(t), \dot{x}(t)\rangle$ for any selection $z(t) \in \partial\varphi(x(t))$.

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In the smooth $\mathbb{R}^d$ case, (CR) holds automatically.

In infinite dimensions, (CR) requires additional structure (e.g. λ-convexity or chain rule conditions of Rossi–Mielke–S.).

Key issues in infinite dimensions

The three-step strategy extends, with additional difficulties:

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1. **Continuity equation in Banach spaces.**

- Test functions: **cylinder functions** $\xi(t, x) = \zeta(t, \langle z_1, x\rangle, \ldots, \langle z_k, x\rangle)$, $\zeta \in C^1_b$.
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2. Narrow topology for Banach-valued measures.

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- Local metrizability and compactness criterion.

3. Hamilton–Jacobi inequality and backward estimates.

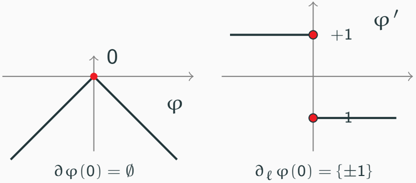
- Cylinder test functions $\Rightarrow$ the HJ constraint involves *finitely many* dual variables.
- The viscosity interpretation of $\partial\varphi$ is crucial: ξ touches φ from below $\Rightarrow D_x\xi \in \partial\varphi(\bar{x})$.

When the subdifferential is not closed

If $\partial\varphi$ is not strong-weak* closed, one uses:

- The **limiting subdifferential** $\partial_\ell\varphi(x) = \{z : \exists\, x_n \to x,\ z_n \in \partial\varphi(x_n),\ z_n \rightharpoonup z\}$.
- The **relaxed slope** $S^-(x) = \inf\left\{ \liminf_n \Psi^*(-z_n) : z_n \in \partial\varphi(x_n),\ x_n \to x \right\}$.

S^- is l.s.c. by construction and $S^- \leqslant S$.



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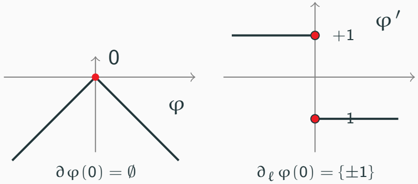

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The full theory applies with S^- replacing S and $\partial_\ell\varphi$ replacing $\partial\varphi$ in the doubly nonlinear equation.

Outline

1. Variational principles for evolution equations
2. De Giorgi's principle and main result
3. Three key arguments
   - Relaxation and continuity equation
   - Von Neumann duality
   - Backward estimates and maximum principle for first order HJ equations
4. The infinite-dimensional setting
5. **Extensions and open problems**

The method naturally extends to **non-autonomous** problems:

$$\partial_v \Psi(t, \boldsymbol{x}(t), \dot{\boldsymbol{x}}(t)) + \partial \varphi(t, \boldsymbol{x}(t)) \ni 0.$$

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When φ depends on time, a further term $\partial_t \varphi(t, \mathbf{x}(t))$ appears in the De Giorgi functional.

Open problems and perspectives

1. **Gradient flows in metric spaces.**

   The De Giorgi functional makes sense in (X, d) metric (via metric derivative and slope).
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3. **Rate-independent systems.**

   The case of 1-homogeneous dissipation Ψ (linear growth) leads to BV solutions and discontinuities — a fundamentally different regime.

Thank you!

Reference: A. Pinzi, F. Riva, G. Savaré, *A direct method for doubly nonlinear equations via convexification in spaces of measures and duality*, preprint 2026.