Existence and regularity properties of spectral optimal partition problems

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Joint works with P. Antunes (ULisboa), Pêdra Andrade (USalzburg), Makson Santos (ULisboa), Ederson Moreira dos Santos (ICMC-USP), Dario Mazzoleni (Pavia), Susanna Terracini (UTorino)

CAMGSD

TÉCNICO LISBOA

The problem

Given $\Omega \subset \mathbb{R}^N$ bounded domain, $k \in \mathbb{N}$ and $0 < a \leq |\Omega|$:

$$\boxed{\inf \left\{ \sum_{i=1}^{k} \lambda_1(\omega_i) \;\middle|\; \begin{array}{c} \omega_1, \ldots, \omega_k \subset \Omega \text{ nonempty open sets} \\ \omega_i \cap \omega_j = \emptyset \text{ for all } i \neq j \text{ and } |\cup_{i=1}^{k} \omega_i| = a \end{array} \right\}}$$

$\lambda_1(\cdot)$ is the first Dirichlet eigenvalue

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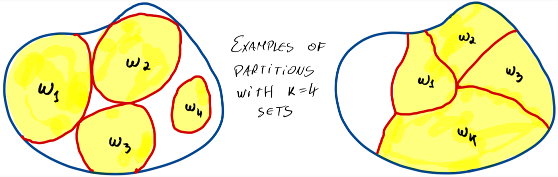

Main results:

- Existence of an optimal solution.
- Optimal regularity of associated eigenfunctions.
- Full characterization of the *free boundary*.

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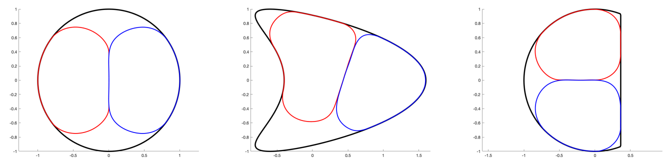

Numerical simulations for $k = 2$ sets. Credit: Pedro Antunes (ULisboa)

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Other (similar) variants: we may consider different energy functionals

$$F(\lambda_1(\omega_1), \dots, \lambda_1(\omega_k)), \qquad F \text{ increasing and coercive in each variable,}$$

or replace $\lambda_1(\omega_i)$ by the energy associated to

$$-\Delta u = f \;\text{ in } \omega_i, \qquad u = 0 \;\text{ on } \partial\omega_i.$$

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Other (different) variants: Restriction on the measure of union of some sets; restriction on the measure of individual components; restriction on the distance between the sets, etc

Outline of the talk:

I. Shape optimization.
A “classical” problem with one shape.

II. Optimal partition problems: statements.

I. Shape optimization.

A “classical” problem with one shape.

Shape Optimization (one shape)

- Class of admissible subsets of Ω: $\mathcal{A}(\Omega)$
- Cost Function: $\Phi : \mathcal{A}(\Omega) \to \mathbb{R}$

Minimization problem:

$$\inf \{\Phi(\omega) : \ \omega_i \in \mathcal{A}(\Omega)\}$$

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Famous Examples:

1. Dido's or isoperimetric problem:

$$\sup \left\{ |A| : \ A \subset \mathbb{R}^N, \ \mathrm{Per}(A) = a \right\}$$

Recommended reading: Dido's Problem and its Impact on Modern Mathematics, by Catherine Bandle, Notices AMS, 2017.

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Famous Examples:

2. Minimizing the fundamental frequency of a drum

$$\inf \left\{ \lambda_1(\omega), \ \ \omega \subset \mathbb{R}^N \text{ nonempty open set, } |\omega| = a \right\}.$$

$\lambda_1(\cdot)$ is the first Dirichlet eigenvalue:

$$-\Delta\varphi = \lambda\varphi \text{ in } \Omega, \qquad \varphi = 0 \text{ on } \partial\Omega.$$

Optimizing frequencies of shapes

$$\boxed{\inf\left\{\lambda_1(\omega),\ \ \omega\subset\mathbb{R}^N \text{ nonempty open set, } |\omega|=a\right\}.}$$

Faber-Krahn inequality $\implies$ solution is a ball

$$|\Omega|^{2/N}\lambda_1(\Omega)\geq|B|^{2/N}\lambda_1(B)$$

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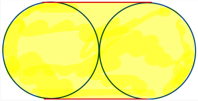
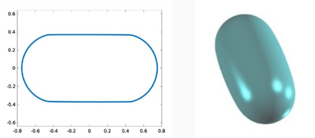
Fun fact: if one asks the shapes to be convex… then Henrot-Oudet (2003) proved that the solution, in $\mathbb{R}^2$, is NOT the stadium ("convex hull of two tangent disks").





Numerical simulations on the right: Antunes-Bogosel, 2022

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What if we restrict the shapes to be inside a “box” Ω?

$$\inf\{\lambda_1(\omega),\ \ \omega\subset\Omega \text{ nonempty open set, } |\omega| = a\} \quad (a < |\Omega|).$$

Measure constraint, no partition

One phase problem with box Ω, a bounded domain:

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[Buttazzo, Dal Maso, 1993]: existence of quasi-open solution (using γ-convergence: works for λ_ℓ, it is the starting point of many papers)

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[Briançon, Hayouni, Pierre, 2005]: existence of open solutions; Lipschitz regularity of associated eigenfunctions

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[T. Briançon and J. Lamboley, 2009]: any open solution ω has a locally finite perimeter and, up to a negligible set, $\partial\omega\cap\Omega$ is analytic.
More accurate results: [Mazzoleni, Terracini, Velichkov 2018]. Regularity up to the fixed boundary [Russ, Trey, Velichkov, 2019].

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$$\inf\left\{\lambda_1(\omega),\ \omega\subset\Omega \text{ nonempty open set},\ |\omega|=a\right\}.$$

Existence of open solutions; Lipschitz regularity of eigenfunctions.

$$\partial\omega\cap\Omega=\mathrm{Reg}\cup\mathrm{Sing},$$

where

$$\mathrm{Reg} \text{ is of class } C^{1,\alpha}, \qquad |\nabla u|=m$$

and Sing has Hausdorff dimension at most $N-5$. More precisely, there is a critical dimension $N^*\in[5,7]$ such that

- if $N<N^*$, then $\mathrm{Sing}=\emptyset$;
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Key starting point

Original problem:

$$\inf\left\{\lambda_1(\omega),\ \omega\subset\Omega \text{ nonempty open set},\ |\omega|=a\right\}.$$

First step: relaxed formulation

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Second step: weaker characterisation in terms of state functions

$$\inf\left\{\frac{\displaystyle\int_\Omega |\nabla u|^2}{\displaystyle\int_\Omega u^2},\ u\in H_0^1(\Omega)\setminus\{0\},\ |\{u\neq 0\}|\leq a\right\}.$$

II. Back to partitions. . .

Optimal Partition Problem

- Class of admissible subsets of Ω: $\mathcal{A}(\Omega)$
- Cost Function: $\Phi : \mathcal{A}(\Omega)^k \to \mathbb{R}$

Minimization problem:

$$\inf \{\Phi(\omega_1, \ldots, \omega_k) : \; \omega_i \in \mathcal{A}(\Omega), \; \omega_i \cap \omega_j = \emptyset \; \forall i \neq j \; + \ldots\}$$

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General questions:

- Does a solution exist?
- What is the regularity of the free boundary: $\cup_i \partial\omega_i$?
- **In some cases:** regularity of associated state functions.

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In general, such a problem does not admit a solution → relaxation (in sense of measures). [Butazzo and Dal Maso (1998)], [Buttazzo and Timofte (2002)]

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Cases in which it is "easier" to have classical solutions
(perhaps in the class of *quasi-open sets*):

- geometric constraints on the admissible domains (convexity, uniform exterior cones, number of connected components of complement, ...)
- Φ has good monotonicity and lower semi-continuity properties

$$\Phi(\omega_1, \ldots, \omega_k) = \sum_{i=1}^{k} \lambda_1(\omega_i).$$

Reference: The book *Variational methods in shape optimization problems, Sections 4 and 5* [Bucur and Buttazzo (2005)].

Statement of our main results

$$\inf\left\{\sum_{i=1}^{k}\lambda_1(\omega_i)\;\middle|\;\begin{array}{c}\omega_i\subset\Omega \text{ are nonempty open sets}\\ \omega_i\cap\omega_j=\emptyset\ \forall i\neq j \text{ and } |\cup_{i=1}^{k}\omega_i|=a\end{array}\right\}\quad(a<|\Omega|)$$

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Theorem [Andrade-Santos-Moreira dos Santos-T., SIAM Math. Anal. 2024]

The problem admits a solution $(\Omega_1, \dots, \Omega_k)$.

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How do the shapes look like?

Theorem (very easy case: Faber-Krahn) For a sufficiently small, any solution is a partition made of k disjoint open balls, all with the same radius.

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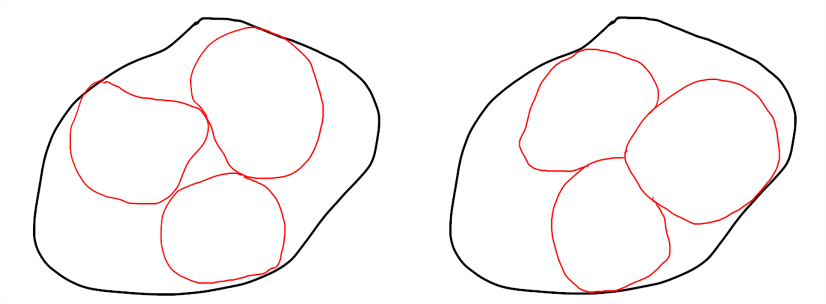
Crucial Question: What happens when k disjoint balls no longer fit inside the domain?

How does a solution look like for k-partition problems, $k \geq 3$?



Any bets?

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Theorem. (Mazzoleni-Santos-T, preprint 2024)

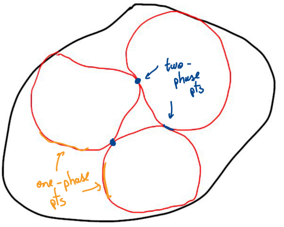
Let $(\Omega_1,\ldots,\Omega_k)$ be **any** optimal partition. Then:

1. **There are no triple** (or higher multiplicity) **points** in Ω, i.e.,

   $(\partial\Omega_i\cap\partial\Omega_j\cap\Omega_k)\cap\Omega=\emptyset$ wherever i,j,k are mutually disjoint.

   Thus, in Ω, free boundary points are either

   two-phase or one-phase.



Recent results

Theorem. (Mazzoleni-Santos-T, preprint 2024)

2. interior two-phase points: let

$$x_0 \in (\partial\Omega_i \cap \partial\Omega_j) \cap \Omega \qquad (i \neq j)$$

   Then, for $\varepsilon \sim 0$, $\partial\Omega_i \cap B_\varepsilon(x_0)$ and $\partial\Omega_j \cap B_\varepsilon(x_0)$ are of class $C^{1,\alpha}$, for some $\alpha \in (0,1]$. Moreover, for associated L^2-normalized eigenfunctions, we have:

$$|\nabla u_i| = |\nabla u_j|, \qquad |\nabla u_i|,\ |\nabla u_j| \geq m > 0$$

Recent results

Theorem. (Mazzoleni-Santos-T, preprint 2024)

3. interior one-phase points:

$$(\partial\Omega_i \cap \Omega) \setminus \bigcup_{\ell \neq i} \partial\Omega_\ell = \mathrm{Reg}_i \cup \mathrm{Sing}_i,$$

where

$$\mathrm{Reg}_i \text{ is of class } C^{1,\alpha}, \qquad |\nabla u_i| = m$$

and Sing_i has Hausdorff dimension at most $N-5$. More precisely, there is a critical dimension $N^* \in [5,7]$ such that

- if $N < N^*$, then $\mathrm{Sing}_i = \emptyset$;
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Recent results

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4. How the free boundary touches $\partial\Omega$: If $\partial\Omega$ Lipschitz, there are **no double** (or higher multiplicity) **points in $\boldsymbol{\partial\Omega}$**, i.e.,

   $$(\partial\Omega_i \cap \partial\Omega_j) \cap \partial\Omega = \emptyset \qquad \text{wherever } i \neq j.$$

   If $\partial\Omega \in C^{1,1}$, then, in a neighborhood of

   $$x_0 \in \partial\Omega_i \cap \partial\Omega,$$

   the set $\partial\Omega_i$ is of class $C^{1,\alpha}$.

What happens when $a \to |\Omega|$?

$$\inf\left\{\sum_{i=1}^{k}\lambda_1(\omega_i)\ \middle|\ \begin{array}{c}\omega_i \subset \Omega \text{ are nonempty open sets for all } i = 1\dots,k,\\ \omega_i \cap \omega_j = \emptyset \text{ for all } i \neq j \text{ and } |\cup_{i=1}^{k}\omega_i| = a\end{array}\right\}$$

Theorem (local uniform bounds in a). As $a \to |\Omega|$, the first eigenfunctions are uniformly bouded in $C^{0,1}_{loc}$.

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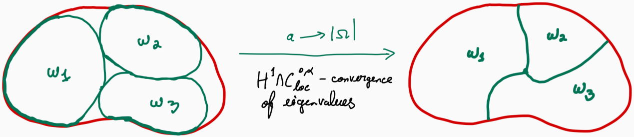
Consequence:



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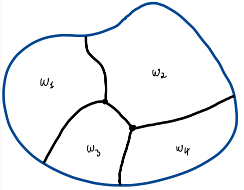

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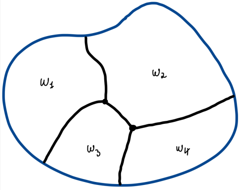
References: Bucur, Buttazzo, Henrot, Helffer, Hoffmann-Ostenhof, Conti, Terracini, T., Verzini, Caffarelli, Lin,. . .



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[Conti-Terracini-Verzini CVPDE '05], complemented by

[Caffarelli-F.H.Lin JAMS '07] and [T.-Terracini CVPDE '12], provide:

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[Caffarelli-F.H.Lin JAMS '07] and [T.-Terracini CVPDE '12], provide:

- Existence of (open) optimal partitions $(\omega_1,\dots,\omega_k)$;
- Global Lipschitz regularity of associated eigenfunctions;
- the inner free boundary $\Omega\cap(\cup_i\partial\omega_i)$ is regular, up to a singular set of Hausdorff dimension at most $N-2$.

Problems without measure constraint

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[Conti-Terracini-Verzini CVPDE '05], complemented by [Caffarelli-F.H.Lin JAMS '07] and [T.-Terracini CVPDE '12], provide:

- Existence of (open) optimal partitions $(\omega_1,\ldots,\omega_k)$;
- Global Lipschitz regularity of associated eigenfunctions;
- the inner free boundary $\Omega\cap(\cup_i\partial\omega_i)$ is regular, up to a singular set of Hausdorff dimension at most $N-2$.

[Alper, CPAM 20]:

- finer results about singular part: locally finite $(N-2)$-Hausdorff measure, and it is $(N-2)$-rectifiable (there is a countable family of regular $(N-2)$-dimensional submanifolds covering the singular set $a.e.-\mathcal{H}^{N-2}$)

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[Ognibene-Velichkov preprint '24]:

- results for the free boundary points that intersect the fixed boundary, $\partial\Omega \cap (\cup_i \partial\omega_i)$: classification, complete understanding of "regular points": intersection is orthogonal. ...

Another fun fact (dimension 2):

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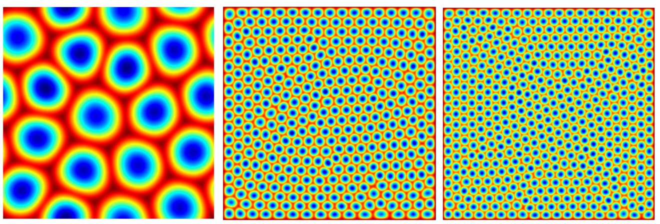
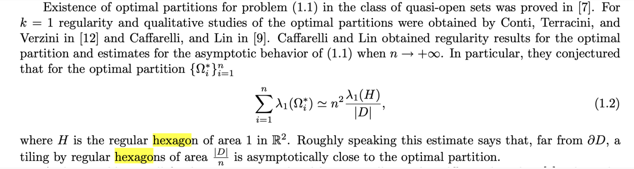
Behavior when $k \to \infty$:



Simulations by Bourdin, Bucur, Oudet (SIAM J. Sci. Comput. 2009)



Existence of optimal partitions for problem (1.1) in the class of quasi-open sets was proved in [7]. For $k = 1$ regularity and qualitative studies of the optimal partitions were obtained by Conti, Terracini, and Verzini in [12] and Caffarelli, and Lin in [9]. Caffarelli and Lin obtained regularity results for the optimal partition and estimates for the asymptotic behavior of (1.1) when $n \to +\infty$. In particular, they conjectured that for the optimal partition $\{\Omega_i^*\}_{i=1}^n$

$$\sum_{i=1}^{n} \lambda_1(\Omega_i^*) \simeq n^2 \frac{\lambda_1(H)}{|D|}, \tag{1.2}$$

where H is the regular hexagon of area 1 in $\mathbb{R}^2$. Roughly speaking this estimate says that, far from ∂D, a tiling by regular hexagons of area $\frac{|D|}{n}$ is asymptotically close to the optimal partition.

Problems without measure constraint: higher eigenvalues

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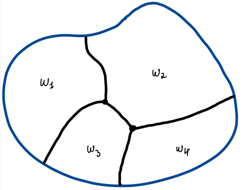
[Ramos-T.-Terracini, ARMA 2016]: results are similar, but the proofs much harder!



Thank you for your attention