Statistical Inference for Poisson-Laguerre tessellation: inversion, reconstruction and stereology

Martina Vittorietti

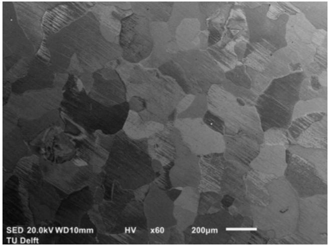
joint work with T. van der Jagt and G. Jongbloed

Delft Institute of Applied Mathematics
TU Delft (NL)

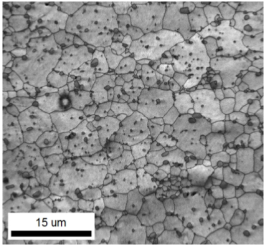
mSPACE conference - WG4

March 16 - March 19, 2026

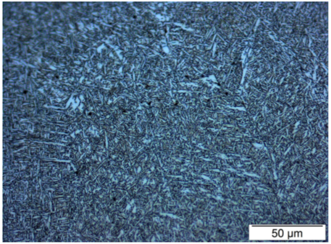
Microstructures



Single-phase steel microstructure

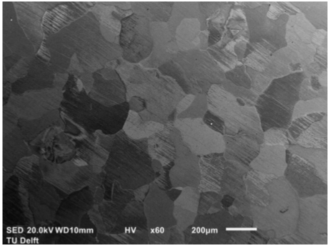


AISI stainless steel with carbides precipitation

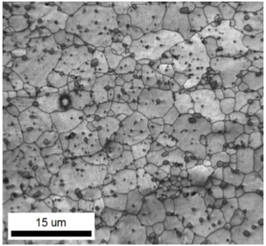


Multi-phase steel microstructure

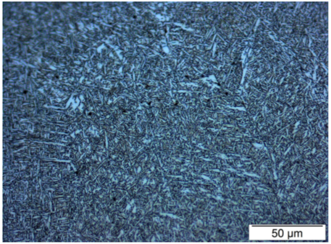
Microstructures



Single-phase steel microstructure



AISI stainless steel with carbides precipitation

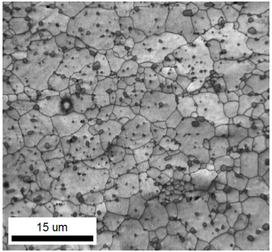


Multi-phase steel microstructure

Microstructure in metallic materials is a three-dimensional arrangement of grains, phases and defects, with all their chemical and structural variety.

Statistical analysis of microstructure features

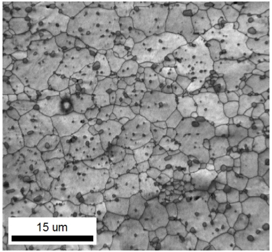
Microstructure features



Statistical analysis of microstructure features

Microstructure features

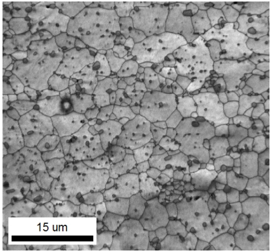
- **grain size**



Statistical analysis of microstructure features

Microstructure features

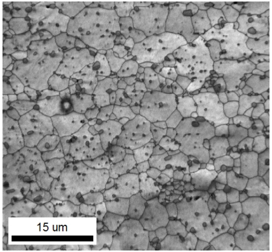
- **grain size**
- grain clustering



Statistical analysis of microstructure features

Microstructure features

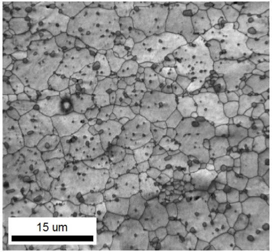
- **grain size**
- grain clustering
- grain orientation



Statistical analysis of microstructure features

Microstructure features

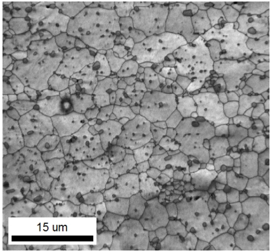
- **grain size**
- grain clustering
- grain orientation
- grain shape



Statistical analysis of microstructure features

Microstructure features

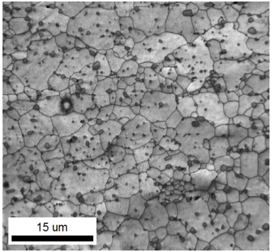
- **grain size**
- grain clustering
- grain orientation
- grain shape
- correlation functions



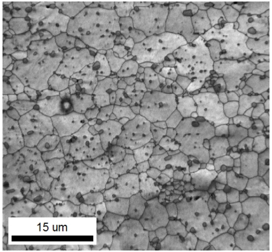
Statistical analysis of microstructure features

Microstructure features

- **grain size**
- grain clustering
- grain orientation
- grain shape
- correlation functions
- tortuosity, constrictivity



Statistical analysis of microstructure features

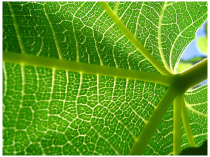


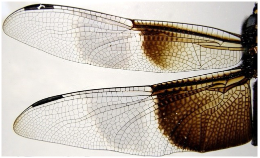
Microstructure features

- **grain size**
- grain clustering
- grain orientation
- grain shape
- correlation functions
- tortuosity, constrictivity

Mechanical properties: hardness, strength, hole expansion capacity, etc.

Poisson-Voronoi Tessellations

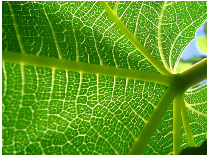


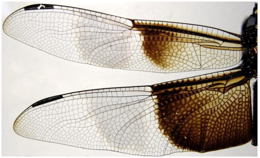


Let $\Phi = \{x_i : i \geq 1\} \subset \mathbb{R}^d$ be a locally finite set of distinct points, called **generators**. The **Voronoi cell** associated with $x_i \in \Phi$ is

$$C(x_i, \Phi) = \{y \in \mathbb{R}^d : \|y - x_i\| \leq \|y - x_j\| \text{ for all } x_j \in \Phi\}.$$

Poisson-Voronoi Tessellations



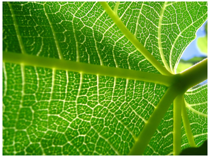


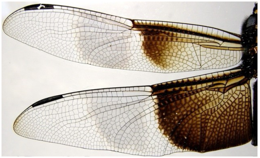
Let $\Phi = \{x_i : i \geq 1\} \subset \mathbb{R}^d$ be a locally finite set of distinct points, called **generators**. The **Voronoi cell** associated with $x_i \in \Phi$ is

$$C(x_i, \Phi) = \{y \in \mathbb{R}^d : \|y - x_i\| \leq \|y - x_j\| \text{ for all } x_j \in \Phi\}.$$

The collection $\mathcal{V}(\Phi) = \{C(x_i, \Phi)\}_{x_i \in \Phi}$ is called the **Voronoi tessellation**.

Poisson-Voronoi Tessellations





Let $\Phi = \{x_i : i \geq 1\} \subset \mathbb{R}^d$ be a locally finite set of distinct points, called **generators**. The **Voronoi cell** associated with $x_i \in \Phi$ is

$$C(x_i, \Phi) = \{y \in \mathbb{R}^d : \|y - x_i\| \leq \|y - x_j\| \text{ for all } x_j \in \Phi\}.$$

The collection $\mathcal{V}(\Phi) = \{C(x_i, \Phi)\}_{x_i \in \Phi}$ is called the **Voronoi tessellation**. If Φ is a homogeneous Poisson point process on $\mathbb{R}^d$ with intensity $\lambda > 0$, we call $\mathcal{V}(\Phi)$ the **Poisson-Voronoi tessellation**.

Poisson-Laguerre tessellation

A **Laguerre tessellation** in $\mathbb{R}^d$ is defined via a set of weighted points $\eta = \{(x_1, h_1), (x_2, h_2), \ldots\}$, called **generators**, where x_i is a point in $\mathbb{R}^d$ and $h_i > 0$.

Poisson-Laguerre tessellation

A **Laguerre tessellation** in $\mathbb{R}^d$ is defined via a set of weighted points $\eta = \{(x_1, h_1), (x_2, h_2), \dots\}$, called **generators**, where x_i is a point in $\mathbb{R}^d$ and $h_i > 0$.
Each generators corresponds to a set, called **cell**:

- empty set
- polytope $\rightarrow$ **tessellation**

Poisson-Laguerre tessellation

A **Laguerre tessellation** in $\mathbb{R}^d$ is defined via a set of weighted points $\eta = \{(x_1, h_1), (x_2, h_2), \ldots\}$, called **generators**, where x_i is a point in $\mathbb{R}^d$ and $h_i > 0$.
Each generators corresponds to a set, called **cell**:

- empty set
- polytope → **tessellation**

Let $\eta^* \subset \eta$ be the **extreme points** of η, the subset of points which generate non-empty cells:

$$\eta^* := \{(x, h) \in \eta : C((x, h), \eta) \neq \emptyset\},$$

where $C((x, h), \eta)$ is the cell associated with (x, h).

Poisson-Laguerre tessellation

A **Laguerre tessellation** in $\mathbb{R}^d$ is defined via a set of weighted points $\eta = \{(x_1, h_1), (x_2, h_2), \dots\}$, called **generators**, where x_i is a point in $\mathbb{R}^d$ and $h_i > 0$.
Each generators corresponds to a set, called **cell**:

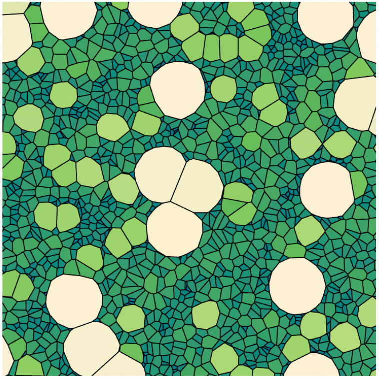
- empty set
- polytope → **tessellation**

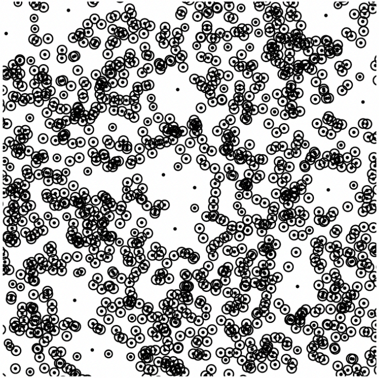
Let $\eta^* \subset \eta$ be the **extreme points** of η, the subset of points which generate non-empty cells:

$$\eta^* := \{(x,h) \in \eta : C((x,h), \eta) \neq \emptyset\},$$

where $C((x,h),\eta)$ is the cell associated with (x,h).
A **Poisson-Laguerre tessellation** is obtained by taking η to be a Poisson point process on $\mathbb{R}^d$ and intensity measure $\nu_d \times \mathbb{F}$, with ν_d a Lebesgue measure on $\mathbb{R}^d$ and $\mathbb{F}$ a non-zero locally finite measure concentrated in $(0, \infty)$.

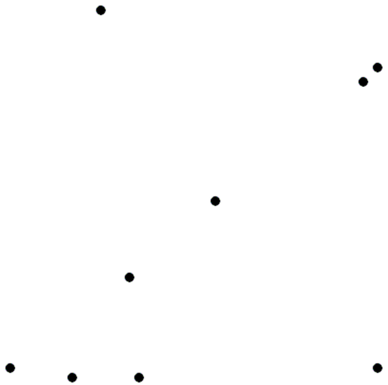
Poisson-Laguerre tessellation

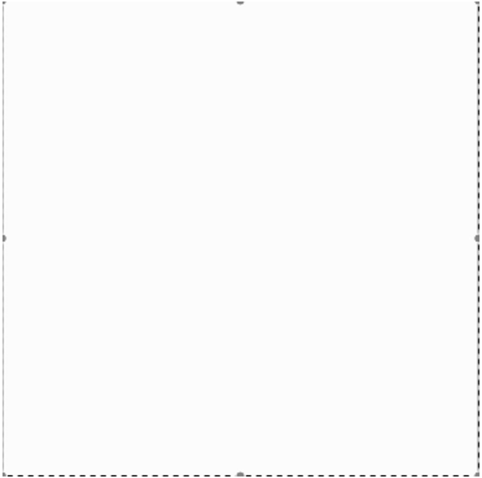




A realization of a planar Poisson-Laguerre tessellation and the corresponding extreme points with circle with radius proportional to the weight of the point.

Voronoi growth comparison





Statistical inference for Poisson-Laguerre tessellation

Effectively, statistical inference for a Poisson-Laguerre tessellation is then reduced to statistical inference for the point process η^*.

Statistical inference for Poisson-Laguerre tessellation

Effectively, statistical inference for a Poisson-Laguerre tessellation is then reduced to statistical inference for the point process η^*.

For $z \geq 0$ we define:

$$F(z) := \mathbb{F}((0, z]).$$

the distribution function of $\mathbb{F}$ $\rightarrow$ this is the only parameter in this model to be estimated.

Statistical inference for Laguerre tessellation

Inversion

Recovering the generator points from the tessellation. (Bourne et al. 2025, Petrich et al. 2019, Quey & Renversade 2018)

Reconstruction

Use a stochastic, nonparametric optimization approach to construct a random tessellation that reproduces selected statistics, e.g., the cell size distribution. (Alpers et al. 2025, Lautensack 2008, Seitl 2022)

Statistical inference for Laguerre tessellation

Inversion

Recovering the generator points from the tessellation. (Bourne et al. 2025, Petrich et al. 2019, Quey & Renversade 2018)

Reconstruction

Use a stochastic, nonparametric optimization approach to construct a random tessellation that reproduces selected statistics, e.g., the cell size distribution. (Alpers et al. 2025, Lautensack 2008, Seitl 2022)

Stereology

Infer three-dimensional characteristics from lower-dimensional observations, such as planar sections. (Gusakova &in Wolde-Lübke 2025, Lautensack & Zuyev 2008)

1. Introduction
2. Voronoi Tessellations
3. Reconstruction
4. Stereology
5. Inversion
6. Conclusions

Inference via a dependent thinning

Let η be a Poisson process on $\mathbb{R}^d \times (0, \infty)$, $d \geq 2$, with intensity measure $\nu_d \times \mathbb{F}$.

Inference via a dependent thinning

Let η be a Poisson process on $\mathbb{R}^d \times (0,\infty)$, $d \geq 2$, with intensity measure $\nu_d \times \mathbb{F}$. Let $y \in \mathbb{R}^d$ and considering the thinning of η:

$$\eta^y := \{(x,h) \in \eta : x + y \in C((x,h),\eta)\}.$$

Note that $\eta^y \subset \eta^* \subset \eta$.

Lemma (van der Jagt, Jongbloed, V., 2025)

Let $B \in \mathscr{B}(\mathbb{R}^d)$, $y \in \mathbb{R}^d$ and $z \geq 0$, the intensity measure Λ^y of η^y satisfies:

$$\Lambda^y(B \times (0,z]) = \nu_d(B) \int_0^z \exp\left(-\kappa_d \int_0^{||y||^2+h} (||y||^2 + h - t)^{\frac{d}{2}} \mathrm{d}F(t)\right) \mathrm{d}F(h).$$

This intensity measure can be computed via the Mecke equation (Mecke, J.(1967)).

Inference via a dependent thinning

Let $\mathscr{F}_+$ denotes the space of all (not necessarily bounded) distribution functions on $(0, \infty)$.

Inference via a dependent thinning

Let $\mathscr{F}_+$ denotes the space of all (not necessarily bounded) distribution functions on $(0,\infty)$.

We define for $F \in \mathscr{F}_+$ the function $G_F : [0,\infty) \to [0,\infty)$ as:

$$G_F(z) := \int_0^z \exp\left(-\kappa_d \int_0^h (h-t)^{\frac{d}{2}} \mathrm{d}F(t)\right) \mathrm{d}F(h).$$

Inference via a dependent thinning

Let $\mathscr{F}_+$ denotes the space of all (not necessarily bounded) distribution functions on $(0, \infty)$.
We define for $F \in \mathscr{F}_+$ the function $G_F : [0, \infty) \to [0, \infty)$ as:

$$G_F(z) := \int_0^z \exp\left(-\kappa_d \int_0^h (h-t)^{\frac{d}{2}} \mathrm{d}F(t)\right) \mathrm{d}F(h).$$

Theorem (*Identifiability*) (van der Jagt, Jongbloed, V., 2025)

Let $F_1, F_2 \in \mathscr{F}_+$, $R > 0$. If $G_{F_1}(z) = G_{F_2}(z)$ for all $z \in [0, R)$ then $F_1(z) = F_2(z)$ for all $z \in [0, R)$,
If $G_{F_1} = G_{F_2}$ then $F_1 = F_2$.

Proof based on a variant of the Grönwall inequality.

First inverse estimator of F

Suppose we observe the extreme points η within the bounded observation window W_n as well as their Laguerre cells and we wish to estimate G_F.
We define the following unbiased estimator for G_F:

$$\hat{G}_n(z) := \frac{1}{\nu_d(W_n)} \sum_{(x,h)\in\eta} \mathbb{1}_{W_n}(x)\mathbb{1}_{(0,z]}(h)\mathbb{1}\{x \in C((x,h),\eta)\}$$
$$= \frac{1}{\nu_d(W_n)} \sum_{(x,h)\in\eta^0} \mathbb{1}_{W_n}(x)\mathbb{1}_{(0,z]}(h),$$

where η^0 represents the point process η^y with $y = 0$.

First inverse estimator of F

Suppose we observe the extreme points η within the bounded observation window W_n as well as their Laguerre cells and we wish to estimate G_F.
We define the following unbiased estimator for G_F:

$$\begin{aligned}\hat{G}_n(z) &:= \frac{1}{\nu_d(W_n)} \sum_{(x,h)\in\eta} \mathbb{1}_{W_n}(x)\mathbb{1}_{(0,z]}(h)\mathbb{1}\{x \in C((x,h),\eta)\} \\ &= \frac{1}{\nu_d(W_n)} \sum_{(x,h)\in\eta^0} \mathbb{1}_{W_n}(x)\mathbb{1}_{(0,z]}(h),\end{aligned}$$

where η^0 represents the point process η^y with $y = 0$.

First inverse estimator of F

Define $\hat{F}_n^0$ to be the unique function $\hat{F}_n^0 \in \mathscr{F}_+$ which satisfies: $G_{\hat{F}_n^0}(z) = \hat{G}_n(z)$ for all $z \geq 0$.

First inverse estimator of F

$\hat{F}_n^0(h_0) = 0$, $\hat{F}_n^0$ is recursively defined via:

$$\hat{F}_n^0(h_i) = \hat{F}_n^0(h_{i-1}) + (\hat{G}_n(h_i) - \hat{G}_n(h_{i-1}) \cdot \exp\left(\kappa_d \sum_{j=1}^{i-1} (h_i - h_j)^{\frac{d}{2}} (\hat{F}_n^0(h_j) - \hat{F}_n^0(h_{j-1}))\right).$$

First inverse estimator of F

$\hat{F}_n^0(h_0) = 0$, $\hat{F}_n^0$ is recursively defined via:

$$\hat{F}_n^0(h_i) = \hat{F}_n^0(h_{i-1}) + (\hat{G}_n(h_i) - \hat{G}_n(h_{i-1}) \cdot \exp\left(\kappa_d \sum_{j=1}^{i-1} (h_i - h_j)^{\frac{d}{2}} (\hat{F}_n^0(h_j) - \hat{F}_n^0(h_{j-1}))\right).$$

For empirical estimators such as $\hat{G}_n$, their consistency follows from a spatial ergodic theorem. See (D. J. Daley and D. Vere-Jones, 2008).

Theorem (*Consistency of* $\hat{F}_n^0$) (van der Jagt, Jongbloed, V., 2025)

With probability one $\lim_{n\to\infty} \hat{F}_n^0(z) = F(z)$ for all $z \geq 0$

First inverse estimator of F

$\hat{F}_n^0(h_0) = 0$, $\hat{F}_n^0$ is recursively defined via:

$$\hat{F}_n^0(h_i) = \hat{F}_n^0(h_{i-1}) + (\hat{G}_n(h_i) - \hat{G}_n(h_{i-1}) \cdot \exp\left(\kappa_d \sum_{j=1}^{i-1} (h_i - h_j)^{\frac{d}{2}} (\hat{F}_n^0(h_j) - \hat{F}_n^0(h_{j-1}))\right).$$

For empirical estimators such as $\hat{G}_n$, their consistency follows from a spatial ergodic theorem. See (D. J. Daley and D. Vere-Jones, 2008).

Theorem (*Consistency of* $\hat{F}_n^0$) (van der Jagt, Jongbloed, V., 2025)

With probability one $\lim_{n\to\infty} \hat{F}_n^0(z) = F(z)$ for all $z \geq 0$

Note this estimator only depends on points (x, h) of η^* with $x \in W_n$ and for which x is located in its own Laguerre cell.

Inference via the volume-biased weight distribution

Volume-biased weight distribution

Let η be a Poisson process on $\mathbb{R}^d \times (0,\infty)$, $d \geq 2$ with intensity measure $\nu_d \times \mathbb{F}$ and $\mathbb{F}$ a locally finite measure concentrated on $(0,\infty)$. Let $A \in \mathscr{B}(\mathbb{R})$, define the following probability measure:

$$\mathbb{F}^V(A) := \mathbb{E}\left(\sum_{(x,h)\in\eta} \mathbb{1}_{[0,1]^d}(x)\mathbb{1}_A(h)\nu_d(C((x,h),\eta))\right).$$

$\mathbb{F}^V$ describes the distribution of the random weight associated with a randomly chosen Laguerre cell, the probability of picking any given cell being proportional to its volume.

Inference via the volume-biased weight distribution

Theorem (van der Jagt, Jongbloed, V., 2025)

Let η be a Poisson process and let $z \geq 0$. Define $F(z) := \mathbb{F}((0,z])$ and $F^V(z) := \mathbb{F}^V((0,z])$, the distribution functions corresponding to $\mathbb{F}$ and $\mathbb{F}^V$ respectively. The measure $\mathbb{F}^V$ is a probability measure and F^V is given by:

$$F^V(z) = 1 - \exp\left(-\kappa_d \int_0^z (z-t)^{\frac{d}{2}} \mathrm{d}F(t)\right) + \frac{d\kappa_d}{2} \int_z^\infty \exp\left(-\kappa_d \int_0^u (u-t)^{\frac{d}{2}} \mathrm{d}F(t)\right) \int_0^z (u-h)^{\frac{d}{2}-1} \mathrm{d}F(h)\mathrm{d}u$$

Corollary (van der Jagt, Jongbloed, V., 2025)

Let $z \geq 0$, if $d = 2$ the CDF F^V is given by:

$$F^V(z) = 1 - \exp\left(-\pi \int_0^z F(t)\mathrm{d}t\right) + \pi F(z) \int_z^\infty \exp\left(-\pi \int_0^u F(t)\mathrm{d}t\right)\mathrm{d}u.$$

Inference via the volume-biased weight distribution

Volume-biased weight distribution induced by F

For $z \geq 0$, $F \in \mathscr{F}_+$ and $m \geq 0$, we define:

$$V(z; F, m) := 1 - \exp\left(-\pi \int_0^z F(t)\mathrm{d}t\right) + \pi F(z)\left(m - \int_0^z \exp\left(-\pi \int_0^u F(t)\mathrm{d}t\right)\mathrm{d}u\right).$$

$$m_F := \int_0^\infty \exp\left(-\pi \int_0^u F(t)\mathrm{d}t\right)\mathrm{d}u.$$

Theorem (van der Jagt, Jongbloed, V., 2025)

Let $F_1, F_2 \in \mathscr{F}_+$, let $R > 0$. If $m_{F_1} = m_{F_2}$ and $V(z; F_1, m_{F_1}) = V(z; F_2, m_{F_2})$ for all $z \in [0, R)$, then $F_1(z) = F_2(z)$ for all $z \in [0, R)$. Consequently, if $m_{F_1} = m_{F_2}$ and $V(\cdot; F_1, m_{F_1}) = V(\cdot; F_2, m_{F_2})$, then $F_1 = F_2$.

Second inverse estimator of F

- We first define an estimator for the distribution function F^V

$$\hat{F}_n^V(z) := \frac{\sum_{(x,h)\in\eta} \mathbb{1}_{W_n}(x)\mathbb{1}_{(0,z]}(h)\nu_d(C((x,h),\eta))}{\sum_{(x,h)\in\eta} \mathbb{1}_{W_n}(x)\nu_d(C((x,h),\eta))}$$

Note that the estimator does not incorporate edge effects.

Second inverse estimator of F

- We first define an estimator for the distribution function F^V

$$\hat{F}_n^V(z) := \frac{\sum_{(x,h)\in\eta} \mathbb{1}_{W_n}(x)\mathbb{1}_{(0,z]}(h)\nu_d(C((x,h),\eta))}{\sum_{(x,h)\in\eta} \mathbb{1}_{W_n}(x)\nu_d(C((x,h),\eta))}$$

  Note that the estimator does not incorporate edge effects.
- We estimate m_F using $\hat{F}_n^0$:

$$\hat{m}_n := m_{\hat{F}_n^0} = \int_0^\infty \exp\left(-\pi\int_0^u \hat{F}_n^0(t)\,\mathrm{d}t\right)\mathrm{d}u.$$

Second inverse estimator of F

- We first define an estimator for the distribution function F^V

$$\hat{F}_n^V(z) := \frac{\sum_{(x,h)\in\eta} \mathbb{1}_{W_n}(x)\mathbb{1}_{(0,z]}(h)\nu_d(C((x,h),\eta))}{\sum_{(x,h)\in\eta} \mathbb{1}_{W_n}(x)\nu_d(C((x,h),\eta))}$$

Note that the estimator does not incorporate edge effects.

- We estimate m_F using $\hat{F}_n^0$:

$$\hat{m}_n := m_{\hat{F}_n^0} = \int_0^\infty \exp\left(-\pi\int_0^u \hat{F}_n^0(t)\mathrm{d}t\right)\mathrm{d}u.$$

Second inverse estimator of F

Define $\hat{F}_n$ to be the unique function $\hat{F}_n \in \mathscr{F}_+$ which satisfies for all $z \geq 0$:

$$\hat{F}_n^V(z) = 1 - \exp\left(-\pi\int_0^z \hat{F}_n(t)\mathrm{d}t\right) + \pi\hat{\mathrm{F}}_\mathrm{n}(\mathrm{z})\left(\hat{m}_n - \int_0^z \exp\left(-\pi\int_0^u \hat{F}_n(t)\mathrm{d}t\right)\mathrm{d}u\right).$$

Consistency

After proving consistency for m_F, we can prove consistency of $\hat{F}_n$

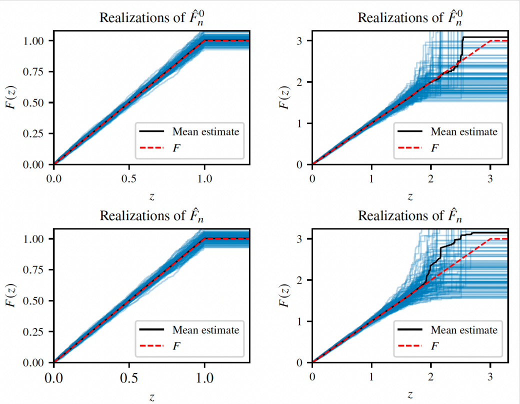
Theorem (Consistency of $\hat{F}_n$) (van der Jagt, Jongbloed, V., 2025)

With probability one, $\lim_{n\to\infty} \hat{F}_n(z) = F(z)$ for all $z \geq 0$.

Simulations

Let $M > 0$ and $z \geq 0$, we consider the following choice for the underlying F:

$$F(z; M) = z \cdot \mathbb{1}\{z < M\} + M \cdot \mathbb{1}\{z \geq M\}$$



M=1 M=3

Sectional Poisson-Laguerre tessellations

Suppose we observe the extreme points and the corresponding cells, of a Poisson-Laguerre tessellation $L(\eta)$ in $\mathbb{R}^{d-1}$, through the observation window W_n.

Sectional Poisson-Laguerre tessellations

Suppose we observe the extreme points and the corresponding cells, of a Poisson-Laguerre tessellation $L(\eta)$ in $\mathbb{R}^{d-1}$, through the observation window W_n.

The underlying Poisson process η has intensity measure $\nu_d \times \mathbb{F}$, where $\mathbb{F}$ is a locally finite measure concentrated on $(0, \infty)$.

This Poisson-Laguerre tessellation in $\mathbb{R}^{d-1}$ is the **sectional tessellation** corresponding to a Poisson-Laguerre tessellation $L(\Psi)$ in $\mathbb{R}^d$.

Sectional Poisson-Laguerre tessellations

Suppose we observe the extreme points and the corresponding cells, of a Poisson-Laguerre tessellation $L(\eta)$ in $\mathbb{R}^{d-1}$, through the observation window W_n.
The underlying Poisson process η has intensity measure $\nu_d \times \mathbb{F}$, where $\mathbb{F}$ is a locally finite measure concentrated on $(0,\infty)$.
This Poisson-Laguerre tessellation in $\mathbb{R}^{d-1}$ is the **sectional tessellation** corresponding to a Poisson-Laguerre tessellation $L(\Psi)$ in $\mathbb{R}^d$.

Intensity measure of Ψ

The Poisson process Ψ of the higher-dimensional tessellation has intensity measure $\nu_d \times \mathbb{H}$, where $\mathbb{H}$ is a locally finite measure concentrated on $(0,\infty)$. For $z \geq 0$ define: $H(z) := \mathbb{H}((0,z])$.

Sectional Poisson-Laguerre tessellations

Suppose we observe the extreme points and the corresponding cells, of a Poisson-Laguerre tessellation $L(\eta)$ in $\mathbb{R}^{d-1}$, through the observation window W_n.
The underlying Poisson process η has intensity measure $\nu_d \times \mathbb{F}$, where $\mathbb{F}$ is a locally finite measure concentrated on $(0,\infty)$.
This Poisson-Laguerre tessellation in $\mathbb{R}^{d-1}$ is the **sectional tessellation** corresponding to a Poisson-Laguerre tessellation $L(\Psi)$ in $\mathbb{R}^d$.

Intensity measure of Ψ

The Poisson process Ψ of the higher-dimensional tessellation has intensity measure $\nu_d \times \mathbb{H}$, where $\mathbb{H}$ is a locally finite measure concentrated on $(0,\infty)$. For $z \geq 0$ define:
$H(z) := \mathbb{H}((0,z])$.

How is H related F?
Can we use a consistent estimator for F to obtain a (locally) consistent estimator for H?

The generator points of the sectional Laguerre diagram

Lemma (van der Jagt, Jongbloed, V., 2025)

Let $\varphi \subset \mathbb{R}^d \times (0,\infty)$ be an at most countable set. Let $\theta \in \mathbb{S}^{d-1}$ and $s \in \mathbb{R}$. Define the hyperplane $T := \{y \in \mathbb{R}^d : \langle \theta, y \rangle = s\}$. For $(x,h) \in \varphi$, with $x \in \mathbb{R}^d$ and $h > 0$ let:

$$x' := x - (\langle \theta, x \rangle - s)\theta$$

$$h' := h + ||x' - x||^2 = h + (\langle \theta, x \rangle - s)^2.$$

Note that $x' \in T$ and define $\varphi' := \{(x', h') : (x,h) \in \varphi\}$. Then, for all $(x,h) \in \varphi$: $C((x,h),\varphi) \cap T = C'((x',h'),\varphi')$ with:

$$C'((x',h'),\varphi') = \{y \in T : ||y - x'||^2 + h' \leq ||y - \bar{x}||^2 + \bar{h} \text{ for all } (\bar{x},\bar{h}) \in \varphi'\}.$$

Preliminaries

- For $x \in \mathbb{R}^d$ write $x = (x_1, x_2, \dots, x_d)$.
- Choose the hyperplane $x_d = 0$ ($\theta = (0, \dots, 0, 1) \in \mathbb{S}^{d-1}$ and $s = 0$)
- Consider the mapping $(x_1, \dots, x_d, h) \mapsto (x_1, \dots, x_{d-1}, 0, h + x_d^2)$.
- Consider the function $\tau : \mathbb{R}^d \times (0, \infty) \to \mathbb{R}^{d-1} \times (0, \infty)$ defined as: $\tau(x, h) = (x_1, \dots, x_{d-1}, h + x_d^2)$;
- $\eta := \tau(\Psi)$ generates the sectional tessellation.
- η is again a Poisson process on $\mathbb{R}^{d-1} \times (0, \infty)$ with intensity measure $\mathbb{E}(\Psi(\tau^{-1}(\cdot)))$.
- Let $B \subset \mathbb{R}^{d-1}$ be a Borel set and $z \geq 0$

$$\tau(x, h) \in B(0, z] \Leftrightarrow x \in B \times \left[-\sqrt{z-h}, \sqrt{z-h}\right] \text{ and } h \leq z.$$

The relationship between F and H

Via the Campbell formula we find:

$$\begin{aligned}\mathbb{E}\Big(\Psi(\tau^{-1}(B\times(0,z]))\Big) &= \int_{\mathbb{R}^d}\int_0^\infty \mathbb{1}\{\tau(x,h)\in B\times(0,z]\}\,\mathrm{d}H(h)\,\mathrm{d}x\\ &= \int_{\mathbb{R}^d}\int_0^\infty \mathbb{1}\Big\{x\in B\times\big[-\sqrt{z-h},\sqrt{z-h}\big]\Big\}\,\mathrm{d}H(h)\,\mathrm{d}x\\ &= \nu_{d-1}(B)\,2\int_0^z\sqrt{z-h}\,\mathrm{d}H(h).\end{aligned}$$

Hence, we obtain

$$F(z) = 2\int_0^z \sqrt{z-h}\,\mathrm{d}H(h)$$

Properties of F and H

- F is an unbounded function, $\lim_{z\to\infty} F(z) = \infty$

Properties of F and H

- F is an unbounded function, $\lim_{z\to\infty} F(z) = \infty$
- F is absolutely continuous and has a Lebesgue density given by:

$$f(z) = \int_0^z \frac{1}{\sqrt{z-t}} \mathrm{d}H(t).$$

Properties of F and H

- F is an unbounded function, $\lim_{z\to\infty} F(z) = \infty$
- F is absolutely continuous and has a Lebesgue density given by:

$$f(z) = \int_0^z \frac{1}{\sqrt{z-t}} \mathrm{d}H(t).$$

- It is possible to express H in terms of F because this is an Abel integral equation.

$$\frac{1}{\pi} \int_0^z \frac{1}{\sqrt{z-t}} \mathrm{d}F(t) = H(z)$$

Estimators for H

A plugin estimator for $H(z)$ is given by:

$$H_n(z) := \frac{1}{\pi} \int_0^z \frac{1}{\sqrt{z-t}} \mathrm{d}\bar{F}_n(t),$$

where $\bar{F}_n$ is a piecewise constant, strongly consistent estimator for F.

Estimators for H

A plugin estimator for $H(z)$ is given by:

$$H_n(z) := \frac{1}{\pi}\int_0^z \frac{1}{\sqrt{z-t}}\mathrm{d}\bar{F}_n(t),$$

where $\bar{F}_n$ is a piecewise constant, strongly consistent estimator for F.
This estimator is, however, rather ill-behaved.

- H_n is not a monotone function;
- H_n is decreasing between jump locations of $\bar{F}_n$;
- if z_0 is a jump location of $\bar{F}_n$ then $\lim_{z\downarrow z_0} H_n(z) = \infty$

Isotonic estimator

- Let $k = k(n)$ be the number of jump locations of $\bar{F}_n$.
- Let $h_1, h_2, \dots, h_k$ with $0 < h_1 < h_2 < \cdots < h_k < \infty$ be the jump locations of $\bar{F}_n$.
- For $z \geq 0$:

$$U_n(z) := \int_0^\infty H_n(t)\,\mathrm{d}t = \frac{2}{\pi}\int_0^z \sqrt{z-t}\,\mathrm{d}\bar{F}_n(t).$$

- Choose (a large) M>0 and define $z_M := \min\{h_k, M\}$.
- Let U_n^M be the greatest convex minorant of U_n on $[0, z_M]$

Isotonic estimator

- Let $k = k(n)$ be the number of jump locations of $\bar{F}_n$.
- Let $h_1, h_2, \ldots, h_k$ with $0 < h_1 < h_2 < \cdots < h_k < \infty$ be the jump locations of $\bar{F}_n$.
- For $z \geq 0$:

$$U_n(z) := \int_0^\infty H_n(t)\mathrm{d}t = \frac{2}{\pi}\int_0^z \sqrt{z-t}\,\mathrm{d}\bar{F}_n(t).$$

- Choose (a large) M>0 and define $z_M := \min\{h_k, M\}$.
- Let U_n^M be the greatest convex minorant of U_n on $[0, z_M]$

Isotonic estimator

$$\hat{H}_n^M(z) := \begin{cases} U_n^{M,r}(z) & \text{if } z \in [0, z_M), \\ U_n^{M,\ell}(z_M) & \text{if } z \geq z_M, \end{cases} \tag{1}$$

where $U_n^{M,\ell}, U_n^{M,r}$ denote the left- and right-derivative of U_n^M.

Consistency of the isotonic estimator

Theorem (*Consistency of* $\hat{H}_n^M$) (van der Jagt, Jongbloed, V.,2025)

Let $M > 0$ and let $\hat{H}_n^M$ be as in (1). Let $z \in [0, M)$, then with probability one:

$$H(z-) \leq \liminf_{n\to\infty} \hat{H}_n^M(z) \leq \limsup_{n\to\infty} \hat{H}_n^M(z) \leq H(z).$$

In particular, if z is a continuity point of H: $\lim_{n\to\infty} \hat{H}_n^M(z) = H(z)$ almost surely.

Consistency of the isotonic estimator

Theorem (*Consistency of $\hat{H}_n^M$*) (van der Jagt, Jongbloed, V.,2025)

Let $M > 0$ and let $\hat{H}_n^M$ be as in (1). Let $z \in [0, M)$, then with probability one:

$$H(z-) \leq \liminf_{n\to\infty} \hat{H}_n^M(z) \leq \limsup_{n\to\infty} \hat{H}_n^M(z) \leq H(z).$$

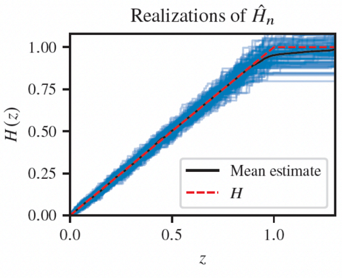
In particular, if z is a continuity point of H: $\lim_{n\to\infty} \hat{H}_n^M(z) = H(z)$ almost surely.

Note that is possible to show that computing the isotonic estimator $\hat{H}_n$ is equivalent to solving an isotonic regression problem.

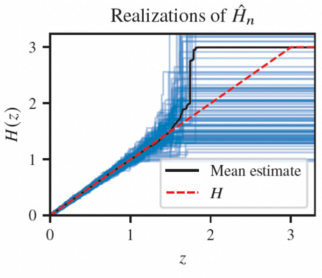
Simulations

Let $M > 0$ and $z \geq 0$, we consider the following choice for the underlying F:

$$F(z; M) = z \cdot \mathbb{1}\{z < M\} + M \cdot \mathbb{1}\{z \geq M\}$$

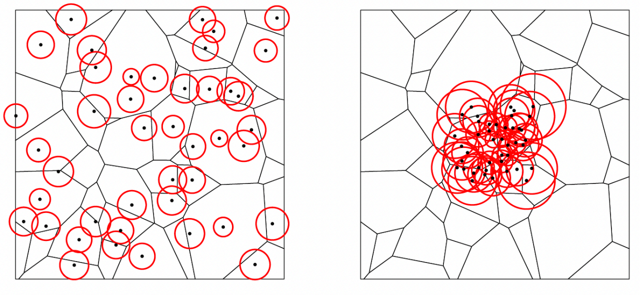


M=1



M=3

Overparametrization



Can we find a characterization of all configurations of weighted points which yield the same Laguerre tessellation?

Overparametrization

Lemma (Meyron, J. (2019))

Let $\mathscr{I} \in \mathbb{N}$ and $\varphi = \{(x_i, h_i)\}_{i\in\mathscr{I}} \subset \mathbb{R}^d \times \mathbb{R}$. Let $\lambda > 0, c \in \mathbb{R}^d$ and $z \in \mathbb{R}$. For $i \in \mathscr{I}$ define:

$$x_i' := \lambda x_i + c$$
$$h_i' = \lambda h_i - \lambda(\lambda - 1)||x_i||^2 - 2\lambda\langle x_i, c\rangle + z$$

Set $\psi = \{x_i', h_i'\}_{i \in \mathscr{I}}$, then $L(\varphi) = L(\psi)$. In fact, $C((x_i, h_i), \varphi) = C((x_i', h_i'), \psi)$ for all $i \in \mathbb{N}$.

Overparametrization

Lemma (Meyron, J. (2019))

Let $\mathcal{I} \in \mathbb{N}$ and $\varphi = \{(x_i, h_i)\}_{i\in\mathcal{I}} \subset \mathbb{R}^d \times \mathbb{R}$. Let $\lambda > 0, c \in \mathbb{R}^d$ and $z \in \mathbb{R}$. For $i \in \mathcal{I}$ define:

$$x_i' := \lambda x_i + c$$
$$h_i' = \lambda h_i - \lambda(\lambda - 1)||x_i||^2 - 2\lambda\langle x_i, c\rangle + z$$

Set $\psi = \{x_i', h_i'\}_i \in \mathcal{I}$, then $L(\varphi) = L(\psi)$. In fact, $C((x_i, h_i), \varphi) = C((x_i', h_i'), \psi)$ for all $i \in \mathbb{N}$.

Theorem

Let $\varphi = \{x_i, h_i)\}_{i\in\mathbb{N}} \subset \mathbb{R}^d \times \mathbb{R}$, $d \geq 2$. Assume φ satisfies the regularity conditions and its points are in general position. Let $\psi \subset \mathbb{R}^d \times \mathbb{R}$ be a countable set of distinct points. Assume that $C((x, h), \varphi) \neq \emptyset$ for each $(x, h) \in \varphi$ and $C((x, h), \psi) \neq \emptyset$ for each $(x, h) \in \psi$. Then, $L(\varphi) = L(\psi)$ if and only if $\psi = \{(x_i', h_i')\}_{i\in\mathbb{N}}$ for some $\lambda > 0$, $c \in \mathbb{R}^d$ and $z \in \mathbb{R}$.

Inverting Poisson-Laguerre tessellations via weighted least-squares

We are in some sense interested in the “best” configuration of the weighted generators, and because all configurations are related to the original configuration, it does not matter which configuration we start with.

Inverting Poisson-Laguerre tessellations via weighted least-squares

We are in some sense interested in the "best" configuration of the weighted generators, and because all configurations are related to the original configuration, it does not matter which configuration we start with.
For all points $(x,h) \in \eta^*$ with $x \in W_n$ we assume that the Laguerre cells corresponding to these points are observed.

Inverting Poisson-Laguerre tessellations via weighted least-squares

We are in some sense interested in the “best” configuration of the weighted generators, and because all configurations are related to the original configuration, it does not matter which configuration we start with.
For all points $(x,h) \in \eta^*$ with $x \in W_n$ we assume that the Laguerre cells corresponding to these points are observed.
We assume also that the original points are known up to a transformation. Specifically, suppose that $\lambda_0 > 0$, $c_0 \in \mathbb{R}^d$ and $f_0 : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d \times \mathbb{R}$ via:

$$f_0(x,h) = \left(\frac{x}{\lambda_0} - \frac{c_0}{\lambda_0}, \frac{1}{\lambda_0} h + \frac{1}{\lambda_0}\left(\frac{1}{\lambda_0} - 1\right)||x||^2 - \frac{2}{\lambda_0}\left\langle x, \frac{c_0}{\lambda_0}\right\rangle\right)$$

Inverting Poisson-Laguerre tessellations via weighted least-squares

We are in some sense interested in the "best" configuration of the weighted generators, and because all configurations are related to the original configuration, it does not matter which configuration we start with.

For all points $(x,h) \in \eta^*$ with $x \in W_n$ we assume that the Laguerre cells corresponding to these points are observed.

We assume also that the original points are known up to a transformation. Specifically, suppose that $\lambda_0 > 0$, $c_0 \in \mathbb{R}^d$ and $f_0 : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d \times \mathbb{R}$ via:

$$f_0(x,h) = \left(\frac{x}{\lambda_0} - \frac{c_0}{\lambda_0}, \frac{1}{\lambda_0}h + \frac{1}{\lambda_0}\left(\frac{1}{\lambda_0} - 1\right)||x||^2 - \frac{2}{\lambda_0}\left\langle x, \frac{c_0}{\lambda_0}\right\rangle\right)$$

For any point $(x,h) \in \eta^*$ with $x \in W_n$ we observe $f_0(x,h)$ instead of (x,h). Note that we assume λ_0 and c_0 to be unknown and we wish to estimate them.

Criterion function

Define $W_n^0 = \frac{1}{\lambda_0} W_n - \frac{c_0}{\lambda_0}$ and a criterion function $T_n : \mathbb{R} \times \mathbb{R}^d \to [0, \infty)$ via:

$$T_n(\lambda, c) := \frac{1}{\nu_d(W_n)} \sum_{(x,h) \in f_0(\eta)} \mathbb{1}_{W_n^0}(x) \int_{C((x,h), f_0(\eta))} ||\lambda x + c - y||^2 \mathrm{d}y$$

$$= \frac{1}{\nu_d(W_n)} \sum_{(x,h) \in \eta} \mathbb{1}_{W_n}(x) \int_{C((x,h),\eta)} \left|\left|\lambda\left(\frac{x}{\lambda_0} - \frac{c_0}{\lambda_0}\right) + c - y\right|\right|^2 \mathrm{d}y$$

and its expectation for $\lambda > 0$ and $c \in \mathbb{R}^d$ is:

$$\mathbb{E}(T_n(\lambda, c)) = \frac{1}{\nu_d(W_n)} \int_{W_n} \left|\left|\left(\frac{\lambda}{\lambda_0} - 1\right)x + c - \frac{\lambda c_0}{\lambda_0}\right|\right| \mathrm{d}x + \int_{\mathbb{R}^d} ||y||^2 p_F(y) \mathrm{d}y.$$

and

$$\mathbb{E}(T_n(\lambda_0, c)) = ||c - c_0||^2 + \int_{\mathbb{R}^d} ||y||^2 p_F(y) \mathrm{d}y.$$

where p_F is a probability density function.

The estimator

The function $(\lambda, c) \mapsto \mathbb{E}(T_n(\lambda, c))$ attains its global minimum in (λ_0, c_0). This inspires the definition of the estimator for λ_0 and c_0:

$$(\hat{\lambda}_n, \hat{c}_n) = \underset{(\lambda,c)\in\mathbb{R}\times\mathbb{R}^d}{\arg\min}\; T_n(\lambda, c)$$

The estimator

The function $(\lambda, c) \mapsto \mathbb{E}(T_n(\lambda, c))$ attains its global minimum in (λ_0, c_0). This inspires the definition of the estimator for λ_0 and c_0:

$$(\hat{\lambda}_n, \hat{c}_n) = \underset{(\lambda,c)\in\mathbb{R}\times\mathbb{R}^d}{\arg\min}\ T_n(\lambda, c)$$

Inversion procedure: for $\lambda \in \mathbb{R}$, $c \in \mathbb{R}^d$, define $f(\cdot;\lambda, c) : \mathbb{R}^{d+1} \to \mathbb{R}^d \times \mathbb{R}$ via:

$$f((x,h);\lambda, c) = (\lambda x + c, \lambda h - \lambda(\lambda - 1)||x||^2 - 2\lambda\langle x, c\rangle).$$

The inversion procedure boils down to computing the following process:

$$\hat{\eta}_n^* := f(f_0(\eta); \hat{\lambda}_n, \hat{c}_n) \cap (W_n \times (0,\infty))$$

The estimator

The function $(\lambda, c) \mapsto \mathbb{E}(T_n(\lambda, c))$ attains its global minimum in (λ_0, c_0). This inspires the definition of the estimator for λ_0 and c_0:

$$(\hat{\lambda}_n, \hat{c}_n) = \underset{(\lambda,c)\in\mathbb{R}\times\mathbb{R}^d}{\arg\min}\ T_n(\lambda, c)$$

Inversion procedure: for $\lambda \in \mathbb{R}$, $c \in \mathbb{R}^d$, define $f(\cdot;\lambda, c) : \mathbb{R}^{d+1} \to \mathbb{R}^d \times \mathbb{R}$ via:

$$f((x,h);\lambda,c) = (\lambda x + c, \lambda h - \lambda(\lambda-1)||x||^2 - 2\lambda\langle x, c\rangle).$$

The inversion procedure boils down to computing the following process:

$$\hat{\eta}^*_n := f(f_0(\eta); \hat{\lambda}_n, \hat{c}_n) \cap (W_n \times (0,\infty))$$

Note that $(\hat{\lambda}_n, \hat{c}_n)$ minimizes the sum of volume weighted squared distances of the generators to the centers of mass of their cells.

Consistency of the inversion procedure

To study the consistency of the estimator $(\hat{\lambda}_n, \hat{c}_n)$, it is essential to study first the behaviour of the following random vector, for $n \in \mathbb{N}$:

$$A_n = \frac{1}{\nu_d(W_n)} \sum_{(x,h)\in\eta} \mathbb{1}_{W_n}(x)\nu_d(C(x,h),\eta)x.$$

Consistency of the inversion procedure

To study the consistency of the estimator $(\hat{\lambda}_n, \hat{c}_n)$, it is essential to study first the behaviour of the following random vector, for $n \in \mathbb{N}$:

$$A_n = \frac{1}{\nu_d(W_n)} \sum_{(x,h)\in\eta} \mathbb{1}_{W_n}(x)\nu_d(C(x,h),\eta)x.$$

A_n may be seen as a weighted average of the generator corresponding to non-empty cells, the weights be the volume of the cells.

Consistency of the inversion procedure

To study the consistency of the estimator $(\hat{\lambda}_n, \hat{c}_n)$, it is essential to study first the behaviour of the following random vector, for $n \in \mathbb{N}$:

$$A_n = \frac{1}{\nu_d(W_n)} \sum_{(x,h)\in\eta} \mathbb{1}_{W_n}(x)\nu_d(C(x,h),\eta)x.$$

A_n may be seen as a weighted average of the generator corresponding to non-empty cells, the weights be the volume of the cells.
The expected values of A_n is the centroid of W_n:

$$\mathbb{E}(A_n) = \frac{1}{\nu_d(W_n)} \int_{W_n} \int_0^\infty \mathbb{E}(\nu_d(C(x,h),\eta))\mathrm{d}F(h)x\mathrm{d}x = \frac{1}{\nu_d(W_n)} \int_{W_n} x\mathrm{d}x = c(W_n).$$

Consistency of the inversion procedure

Theorem (*Consistent inversion*) (van der Jaght, Jongbloed, V.,2026)

1. If the sequence $(A_n)_{n\geq 1}$ is uniformly tight, then:

$$\lim_{n\to\infty}(\hat{\lambda}_n\hat{c}_n) \stackrel{\mathbb{P}}{=} (\lambda_0, c_0).$$

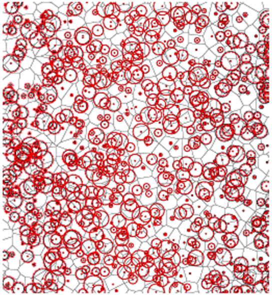
2. If the random variable $\sup_{n\geq 1}||A_n||$ is almost surely finite, then:

$$\lim_{n\to\infty}(\hat{\lambda}_n\hat{c}_n) \doteq (\lambda_0, c_0).$$

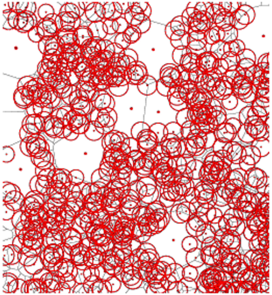
Simulations

Given that when a Poisson-Laguerre tessellation is observed through a window W_n, the cells at the boundary are only partially observed, so we consider another estimator for (λ_0, c_0) :

$$(\bar{\lambda}_n, \bar{c}_n) = \underset{(\lambda,c)\in\mathbb{R}\times\mathbb{R}^d}{\arg\min}\ \nu_d(W_n) \sum_{(x,h)\in f_0(\eta)} \mathbb{1}\{C((x,h), f_0(\eta)) \subset W_n\} \nu_{x,h} ||\lambda x + c - c_{x,h}||^2.$$



$F_1(z) = z \cdot \mathbb{1}\{z < 1\} + \mathbb{1}\{z \geq 1\}$



$F_2(z) = 0.01 \cdot \mathbb{1}\{z \geq 1\} + 0.04 \cdot \mathbb{1}\{z \geq 8\} + 0.95 \cdot \mathbb{1}\{z \geq 10\}$

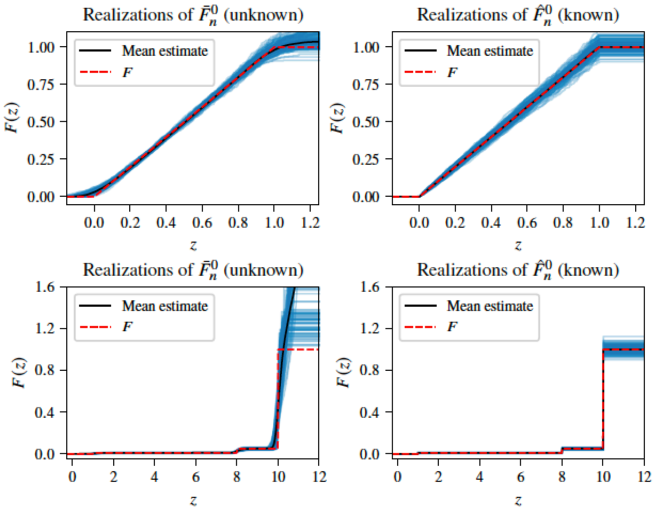
Estimation of F

Let $\hat{\eta}_n^*$ denote the configuration of weighted generators obtained via the inversion procedure defined via $(\bar{\lambda}_n, \bar{c}_n)$. For $z \geq 0$ define:

$$\bar{G}_n(z) := \frac{1}{\nu_d(W_n)} \sum_{(\hat{x},\hat{h}) \in \hat{\eta}_n^*} \mathbb{1}_{W_n}(\hat{x}) \mathbb{1}_{(0,z]}(\hat{h}) \mathbb{1}\{\hat{x} \in C((\hat{x},\hat{h}), \hat{\eta}_n^*)\}$$

$$\bar{F}_n^0(\hat{h}_i) = \bar{F}_n^0(\hat{h}_{i-1}) + (\bar{G}_n(\hat{h}_i) - \bar{G}_n(\hat{h}_{i-1})) \cdot \exp\left(\kappa_d \sum_{j=1}^{i-1} (\hat{h}_i - \hat{h}_j)^{\frac{d}{2}} (\bar{F}_n^0(\hat{h}_j) - \bar{F}_n^0(\hat{h}_{j-1})\right).$$

Simulations



1. Introduction
2. Voronoi Tessellations
3. Reconstruction
4. Stereology
5. Inversion
6. **Conclusions**

Conclusions & Future Directions

- Consistency of the estimators based on weighted generators obtained via the inversion procedure are in general consistent is an important open problem.

Conclusions & Future Directions

- Consistency of the estimators based on weighted generators obtained via the inversion procedure are in general consistent is an important open problem.
- Edge effects

Conclusions & Future Directions

- Consistency of the estimators based on weighted generators obtained via the inversion procedure are in general consistent is an important open problem.
- Edge effects
- Which estimator?

Conclusions & Future Directions

- Consistency of the estimators based on weighted generators obtained via the inversion procedure are in general consistent is an important open problem.
- Edge effects
- Which estimator?
- Asymptotics

Conclusions & Future Directions

- Consistency of the estimators based on weighted generators obtained via the inversion procedure are in general consistent is an important open problem.
- Edge effects
- Which estimator?
- Asymptotics
- Stereology

Conclusions & Future Directions

- Consistency of the estimators based on weighted generators obtained via the inversion procedure are in general consistent is an important open problem.
- Edge effects
- Which estimator?
- Asymptotics
- Stereology
- Application to real data

References I

Alpers, A., Furat, O., Jung, C., Neumann, M., Redenbach, C., Saken, A., & Schmidt, V. (2025). Comparative Analysis of Algorithms for the Fitting of Tessellations to 3D Image Data. arXiv preprint arXiv:2507.14268.

Bourne, D. P., Pearce, M., & Roper, S. M. (2025). Inverting Laguerre tessellations: recovering tessellations from the volumes and centroids of their cells using optimal transport. ESAIM: Mathematical Modelling and Numerical Analysis, 59(2), 841-871.

Bourne, D. P., Gallouët, T. O., Merigot, Q., & Natale, A. (2025). Semi-discrete convex order and Laguerre tessellation fitting.

Daley, D. J., & Vere-Jones, D. (2008). An introduction to the theory of point processes: volume II: general theory and structure. New York, NY: Springer New York.

Gusakova, A., & in Wolde-Lübke, M. (2025). Poisson-Laguerre tessellations. Electronic Journal of Probability, 30, 1-48.

van Der Jagt, T., Jongbloed, G., & Vittorietti, M. (2025). Nonparametric inference for Poisson-Laguerre tessellations. Scandinavian Journal of Statistics, 52(4), 1816-1851.

van Der Jagt, T., Jongbloed, G., & Vittorietti, M. (2026). Inverting Poisson-Laguerre tessellations.(*in preparation*).

References II

Lautensack, C., & Zuyev, S. (2008). Random laguerre tessellations. Advances in applied probability, 40(3), 630-650.

Lautensack, C. (2008). Fitting three-dimensional Laguerre tessellations to foam structures. Journal of Applied Statistics, 35(9), 985-995.

Meyron, J. (2019). Initialization procedures for discrete and semi-discrete optimal transport. Computer-Aided Design, 115, 13-22.

Petrich, L., Staněk, J., Wang, M., Westhoff, D., Heller, L., Šittner, P., . . . & Schmidt, V. (2019). Reconstruction of grains in polycrystalline materials from incomplete data using Laguerre tessellations. Microscopy and Microanalysis, 25(3), 743-752.

Quey, R., & Renversade, L. (2018). Optimal polyhedral description of 3D polycrystals: Method and application to statistical and synchrotron X-ray diffraction data. Computer Methods in Applied Mechanics and Engineering, 330, 308-333.

Seitl, F., Møller, J., & Beneš, V. (2022). Fitting three-dimensional Laguerre tessellations by hierarchical marked point process models. Spatial Statistics, 51, 100658.

Thank you for listening !

Martina Vittorietti

m.vittorietti@tudelft.nl