Continuity aspects for traces of Dirichlet forms w.r.t. monotone weak convergence of measures

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Abstract

Traces of Dirichlet forms are quadratic forms associated with boundary operators. In these notes we prove lower semi continuity of spectra, convergence of ordered eigenvalues and eigenfunctions, for traces of Dirichlet forms when the speed measure is the monotone weak limit of G-Kato measures.

1. Introduction

Given a regular Dirichlet form $\mathcal{E}$ with domain in $L^2(X,m)$ and another measure μ, let $\mathcal{E}^\mu$ be the trace of $\mathcal{E}$ with respect to μ, which is again a Dirichlet from with domain in $L^2(X,\mu)$. We study continuous dependence of spectral objects with respect to variations of the underlying measure μ. Precisely, we show that if $(\mu_n)_{n\in\mathbb{N}\cup\{\infty\}}$ is a family of finite G-Kato measures such that $\mu_n\uparrow\mu_\infty$ or $\mu_n\downarrow\mu_\infty$ weakly, then spectral objects (such as eigenvalues, Riesz eigenprojections, eigenvectors) related to $\mathcal{E}^{\mu_\infty}$ can be approximated by the corresponding spectral objects of the sequence $(\mu_n)_n$. Furthermore, we will show that for large integers n, the dimensions of the spectral projections are locally constant, obtaining thereby convergence of the ordered eigenvalues and hence of Rayleigh–Ritz quotients.
Let us stress that the trace forms $\mathcal{E}^{\mu_n}$ have domains in varying spaces, namely $L^2(X,\mu_n)$, so that the above-mentioned convergence problems become delicate to handle. Though there is a method for discussing the problem by means of Mosco convergence on varying Hilbert spaces (see [6, 3]); we give here a different method based on tools from potential theory of Dirichlet forms.
Our results recover the one-dimensional case studied in [5]. We quote that traces of Dirichlet forms are also called time-changed Dirichlet forms in the literature [4]. In fact the semigroup related to $\mathcal{E}^\mu$ is obtained from the semigroup of the starting form $\mathcal{E}$ by an appropriate time change. This leads to a probabilistic interpretation of boundary operators.
This poster is part from the paper [1] which will appear in Kyoto J.Math.

2. Framework

Let X be a locally compact separable metric space, m a positive Radon measure on Borel subset of X with full support and $\mathcal{E}$ a regular symmetric Dirichlet form with domain $\mathcal{D}\subset L^2(X,m)$. We denote by $\|\cdot\|$ the $L^2(X,m)$ norm and set $\mathcal{E}[u]:=\mathcal{E}(u,u)$. Let μ be an other positive Radon measure on Borel subsets of X with no sets having zero capacity (smooth measure). Set $\mathscr{B}_b(X)$ resp. $\mathscr{B}_b^+(X)$ the spaces of Borel bounded and positive Borel bounded functions on X endowed with the topology of uniform convergence.
To avoid technicalities, we assume that $\mathcal{E}$ is transient (this assumption is not restrictive. For, if $\mathcal{E}$ were not transient one changes $\mathcal{E}$ by $\mathcal{E}_\alpha:=\mathcal{E}+\alpha\|\cdot\|^2$ for some $\alpha>0$ which is always transient). Let $\mathcal{D}^e$ be the extended space of $\mathcal{E}$ and $\mathcal{E}^e$ the extension of $\mathcal{E}$ on $\mathcal{D}^e$. Then $(\mathcal{D}^e,\mathcal{E}^e)$ is a Hilbert space; moreover elements from $\mathcal{D}^e$ have m-representatives which are quasi continuous (q.c. for short) and two q.c. representatives coincide quasi everywhere (q.e. for short) and hence μ a.e., by assumption on μ. We assume that elements from $\mathcal{D}^e$ have been chosen to be q.c.
We assume that the following Hardy-type inequality holds: there is a constant $C_H>0$ such that

$$\int_X u^2\,d\mu\le C_H\mathcal{E}^e[u],\ u\in\mathcal{D}^e. \tag{2.1}$$

Smoothness of μ in conjunction with Hardy's inequality imply that the linear operator

$$J:\mathrm{dom}(J)=(\mathcal{D}^e,\mathcal{E}^e)\to L^2(X,\mu),\ u\mapsto u$$

is well defined, bounded and has dense range.
Let P^e be the $\mathcal{E}^e$ orthogonal projection onto the $\mathcal{E}^e$-orthogonal of $\ker(J)$ and $\mathcal{E}^\mu$ be the quadratic form defined by

$$\mathrm{dom}(\mathcal{E}^\mu)=\mathrm{ran}(J),\mathcal{E}^\mu[Ju]=\mathcal{E}^e[P^eu],\ u\in\mathrm{ran}(J).$$

Then $\mathcal{E}^\mu$ is a Dirichlet form (see [2, theorem 5.2]). It is called the trace of $\mathcal{E}$ w.r.t. the measure μ.
Let $(R_\alpha^\mu)_{\alpha>0}$ be the resolvent family of $\mathcal{E}^\mu$ and L_μ be the positive self-adjoint operator related to $\mathcal{E}^\mu$ via Kato representation theorem and $(\mathbf{P}_t)_{t>0}$ the semigroup associated with $\mathcal{E}$.
From now on we assume that there is a jointly measurable symmetric function

$$p_t(\cdot,\cdot):(0,\infty)\times X\times X\to(0,\infty)$$

such that

$$\mathbf{P}_tu(x)=\int_X p_t(x,y)u(y)\,dm(y),\ t>0,x\in X,u\in L^2(X,m).$$

For any $\alpha\ge0$ we designate by G_α the α-order resolvent kernel of $\mathcal{E}$:

$$G_\alpha(x,y):=\int_0^\infty e^{-\alpha t}p_t(x,y)\,dt,\ x,y\in X. \tag{2.2}$$

Let L be the positive self-adjoint operator associated with $\mathcal{E}$. Then for any $\alpha>0,u\in L^2(X,m)$ we have

$$(L+\alpha)^{-1}u=\int_X G_\alpha(\cdot,y)u(y)\,dm(y),m-a.e..$$

For $\alpha=0$ we set $G:=G_0$. By transience of $\mathcal{E}$ we have $G_\alpha\not\equiv\infty$, moreover $G_\alpha>0$ for any $\alpha\ge0$.
We further assume that, for every fixed $y\in X$ the function

$$G(\cdot,y):X\to(0,\infty]\text{ is lower semi-continuous.}$$

Consider

$$K^\mu:L^2(X,\mu)\to L^2(X,\mu),\ u\mapsto\int_X G(\cdot,y)u(y)\,d\mu(y)$$

Hardy's inequality implies (and is in fact equivalent) that K^μ is bounded.

Definition 2.1. Let μ be a positive Radon measure on X. We say that μ is a G-Kato measure if

$$G^\mu 1:=\int_X G(\cdot,y)\,d\mu(y)\in C_0(X).$$

We mention that every G-Kato measure satisfies Hardy's inequality.

3. Main results

Proposition 3.1. *The following assertions hold:*

1. *A resolvent formula:*
$$R_\alpha^\mu=(1+\alpha K^\mu)^{-1}K^\mu,\ \alpha>0. \tag{3.1}$$
2. *$\mathcal{E}^\mu$ has compact resolvent if and only if K^μ is compact.*
3. *All eigenvalues of $\mathcal{E}^\mu$ are strictly positive. Moreover E is an eigenvalue of $\mathcal{E}^\mu$ if and only $1/E$ is an eigenvalue of K^μ with the same eigenfunction and the same multiplicity.*

Let μ be a finite G-Kato measure, consider

$$G^\mu:\mathscr{B}_b(X)\to\mathscr{B}_b(X),\ u\mapsto\int_X G(\cdot,y)u(y)\,d\mu(y).$$

Lemma 3.2. *Let μ be a finite G-Kato measure. Then K^μ,G^μ and $K^\mu|_{L^\infty(X,\mu)}$ are compact. They all have the same eigenvalues and the same eigenfunctions.*

Prop.3.1 and Lemma 3.2 imply that the spectrum of $\mathcal{E}^\mu$ consists of eigenvalues with finite multiplicities, possibly tending to ∞, which we arrange in an increasing way, repeated as many times as their respective multiplicities.
Let $(\mu_n)_{n\mathbb{N}\cup\{\infty\}}$ be a family of finite G-Kato measures such that $\mu_n\to\mu_\infty$ weakly-monotonically, i.e.,

$$\text{either}\int_X\varphi\,d\mu_n\uparrow\int_X\varphi\,d\mu_\infty\text{ for all }\varphi\in C_b^+(X)$$
$$\text{or}\int_X\varphi\,d\mu_n\downarrow\int_X\varphi\,d\mu_\infty\text{ for all }\varphi\in C_b^+(X).$$

Observe that $\mathrm{dom}(\mathcal{E}^{\mu_n})\subset L^2(X,\mu_n)$ for each n, which are varying Hilbert spaces. Here is a first aspect of continuity.

Theorem 3.3. *Let $(\mu_n)_n$ be as above, $\lambda>0$. For any interval $I\subset\mathbb{R}$ let $P_n(I)$ be the spectral projection with respect to $\mathcal{E}^{\mu_n}$ and $H_n:=L^2(X,\mu_n)$, $n\in\mathbb{N}\cup\{\infty\}$. Then for large n it holds*

$$\dim P_n(-\infty,\lambda)H_n=\dim P_\infty(-\infty,\lambda)H_\infty.$$

For each $n\in\mathbb{N}\cup\{\infty\}$ let $E_n^{(k)}$ be the enumeration of the eigenvalues of $\mathcal{E}^{\mu_n}$:

$$0<E_n^{(1)}\le E_n^{(2)}\le\cdots\le E_n^{(k)}\le\cdots(\to\infty).$$

Theorem 3.4. *Let $(\mu_n)_n$ be as above. Then*

1. *Let E_∞ be an eigenvalue of $\mathcal{E}^{\mu_\infty}$ of multiplicity m. Then for large n there is m eigenvalues (counted as many times as their respective multiplicities) $\tilde{E}_n^{(1)},\cdots\tilde{E}_n^{(m)}$ of $\mathcal{E}^{\mu_n}$ such that $\lim_{n\to\infty}\tilde{E}_n^{(j)}=E_\infty$, $j=1,\cdots m$.*
2. *For every $\mathscr{B}_b(X)$-normalized eigenfunction $(v_n^{(j)})$ of $\tilde{E}_n^{(j)}$ there is a subsequence $(v_{n_k}^{(j)})$ which converges uniformly to an eigenvector of E_∞.*
3. *$\lim_{n\to\infty}E_n^{(k)}=E_\infty^{(k)}$ for all k.*

Remark 3.5. Theorem 3.4 implies convergence of the inf-sup involved in Rayleigh-Ritz variational formula for ordered eigenvalues. Precisely, we have

$$\lim_{n\to\infty}\inf_{\dim\mathscr{L}=k}\sup\left\{\mathcal{E}^{\mu_n}[u],u\in\mathscr{L}\subset\mathrm{dom}(\mathcal{E}^{\mu_n}),\|u\|_{\mu_n}=1\right\}=\inf_{\dim\mathscr{L}=k}\sup\left\{\mathcal{E}^{\mu_\infty}[u],u\in\mathscr{L}\subset\mathrm{dom}(\mathcal{E}^{\mu_\infty}),\|u\|_{\mu_\infty}=1\right\}.$$

Example 3.6. Let $X=\mathbb{R},m=dx$ the Lebesgue measure on $\mathbb{R}$ and

$$\mathrm{dom}(\mathcal{E})=H^1(\mathbb{R}),\mathcal{E}[u]=\int_{\mathbb{R}}(u'(x))^2\,dx+\int_{\mathbb{R}}u^2(x)\,dx.$$

In this case $G(x,y)=\frac{1}{2}e^{-|x-y|},x,y\in\mathbb{R}$. Let $(a_k)_{k\in\mathbb{Z}}\subset(0,\infty)$ be such that $\sum_k a_k<\infty$ and $\mu_n=\sum_{|k|\le n}a_k\delta_k$. Then $\mu_n\uparrow\mu_\infty$. By Sobolev inequality each μ_n satisfies Hardy's inequality. We also have $(\mu_n)_n\subset\mathscr{K}_b(\mathbb{R})$. Set F_n the support of μ_n. Then

$$J_n:H^1(\mathbb{R})\to L^2(\mathbb{R},\mu_),\ u\mapsto u|_{F_n},$$
$$K^{\mu_n}u=\frac{1}{2}\int_{\mathbb{R}}e^{-|\cdot-y|}u(y)\,d\mu_n(y),u\in L^2(\mathbb{R},\mu_n),n\in\mathbb{N}_\infty.$$

Thus the J_n's have dense range and K^{μ_n}'s are compact and hence $\mathcal{E}^{\mu_n}$'s have compact resolvent. Following [2, Example 4.3] we obtain: For $n\in\mathbb{N}$, $\mathrm{dom}(\mathcal{E}^{\mu_n})=\mathbb{R}^{2n+1}$ and

$$\mathcal{E}^{\mu_n}(\psi,\psi)=\frac{1}{\sinh 1}\sum_{|k|\le n-1}(\psi_{(k+1)}-\psi_k)^2+2\frac{\cosh 1-1}{\sinh 1}\sum_{|k|\le n-1}\psi_k^2+(\psi_{-n}^2+\psi_n^2).$$

Whereas $\mathrm{dom}(\mathcal{E}^{\mu_\infty})=\{\psi\in L^2(\mathbb{R},\mu):\sum_{n\in\mathbb{Z}}|\psi(n)|^2<\infty\}$,

$$\mathcal{E}^{\mu_\infty}[\psi]=\frac{1}{\sinh 1}\sum_{k\in\mathbb{Z}}(\psi_{k+1}-\psi_k)^2+2\frac{\cosh 1-1}{\sinh 1}\sum_{k\in\mathbb{Z}}\psi_k^2.$$

Let us quote that for $n\in\mathbb{N}$ the form $\mathcal{E}^{\mu_n}$ has finitely many eigenvalues, whereas $\mathcal{E}^{\mu_\infty}$ has infinitely many distinct eigenvalues.
In this case the conclusions of Theorems 3.3 and 3.4 hold.

References

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