Ambarzumian Theorem for Quantum Graphs with Magnetic Potential

Alice Brolin

We consider a compact **metric graph** $\Gamma = (\mathbf{E}, \mathbf{V})$

- Edges $e_1, ..., e_N$, $e_n = [x_{2n-1}, x_{2n}]$ and $\ell_n = x_{2n} - x_{2n-1}$.
- Vertices $v_1, ..., v_M$ are a partition of the endpoints $x_1, ..., x_{2N}$
- The total length is $\ell(\Gamma) = \sum_{n=1}^{N} \ell_n$
- $L^2(\Gamma) = \bigoplus_{n=1}^{N} L^2(e_n)$ Hillbert space of functions on Γ.

Take $a \in \bigoplus C^1(e_n)$, $q \in \bigoplus L^\infty(e_n)$. The **Magnetic Schrödinger operator** $L^{st}_{q,a}$ acts as

$$-\left(\frac{d}{dx} - ia\right)^2 + q$$

Let $\partial u(x_j) = (-1)^{j+1} \lim\limits_{x \to x_j} (u'(x) - ia(x)u(x))$. Then $dom(L^{st}_{q,a})$ consists of $W^{2,2}$ functions with **Standard conditions:**

$$\begin{cases} u(x_{j_1}) = u(x_{j_2}),\ x_{j_1}, x_{j_2} \in v_m \\ \sum_{x_j \in v_m} \partial u(x_j) = 0 \end{cases}$$

We have an **Ambarzumian theorem** for Schrödinger operators on metric graphs

Theorem (Davies, 2013)**.** *If the lowest eigenvalue* $\lambda_1(L^{st}_q) \geq 0$ *and* $\limsup\limits_{n\to\infty} \lambda_n(L^{st}_q) - \lambda_n(L^{st}_0) \leq 0$ *then* $q \equiv 0$

The magnetic potential

Let Θ be a function s.t. $(\Theta|_{e_n})' = a|_{e_n}$.

Then $\left(\frac{d}{dx} - ia\right) e^{i\Theta} = e^{i\Theta} \frac{d}{dx}$

$L^\Theta_q := e^{-i\Theta} L^{st}_{q,a} e^{i\Theta}$ acts, on each edge, as $-\dfrac{d^2}{dx^2} + q$ with vertex conditions

$$\begin{cases} e^{i\Theta(x_{j_1})} u(x_{j_1}) = e^{i\Theta(x_{j_2})} u(x_{j_2}), & x_{j_1}, x_{j_2} \in v^m \\ \displaystyle\sum_{x_j \in v_m} e^{i\Theta(x_j)} \partial u(x_j) = 0, & x_j \in v_m \end{cases}$$

where $\partial u(x_j) = (-1)^{j+1} \lim\limits_{x \to x_j} u'(x)$

If $L^\Theta_q = L^{st}_q$ then we say that the magnetic potential is **globally eliminated**. The following are equivalent:

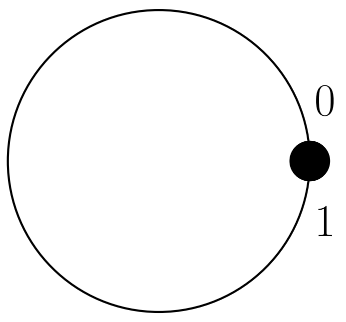
- The magnetic potential can be globally eliminated
- $e^{i\Theta}$ is continuous on Γ for some Θ
- For any cycle C: $\oint_C a dx \in 2\pi\mathbb{Z}$

For a magnetic Laplacian $L^{st}_{0,a}$ the lowest eigenvalue is given by

$$\lambda_1(L^{st}_{0,a}) = \min_{\substack{u \in C(\Gamma) \\ u \in W^{1,2}}} \frac{\int_\Gamma |u' - iau|^2 dx}{\int_\Gamma |u|^2 dx},$$

Proposition. $\lambda_1(L^{st}_{0,a}) = 0$ *if and only if the magnetic potential can be globally eliminated*

Proof. We have continuous u such that $\int_\Gamma |u' - iau|^2 dx = 0$. Then u must be of the form $u = e^{i\Theta}$ where $\Theta' = a$. ☐



Example Let $L^{st}_{0,a}$ be an operator on a loop Γ consisting of one edge $[0,1]$. Let $\Theta(x) = \int_0^x a(t)dt$.

- The eigenvalues of $L^{st}_{0,a}$ are $(2\pi n + \Theta(1))^2$ where $n \in \mathbb{Z}$.
- $L^{st}_{0,a}$ is isospectral to $L^{st}_{0,0}$ if and only if $\Theta(1) = 2\pi n$ for $n \in \mathbb{Z}$
- If $\Theta(1) = 2\pi n$ then $e^{i\Theta(x)} \in C(\Gamma)$ and $e^{-i\Theta} L^{st}_{0,a} e^{i\Theta} = L^{st}_{0,0}$

Reference Laplacians

The reference Laplacian of a Schrödinger operator L_q (with arbitrary vertex conditions) on Γ is the operator with quadratic form $Q(u) = \int_\Gamma |u'|^2 dx$, $dom(Q) = \overline{dom(L_q)}$.

Theorem(Kurasov, Suhr, 2020)**.** *If* L_{q_1}, L_{q_2} *are two Schrödinger operators on* Γ_1 *and* Γ_2 *respectively and*

$$\sqrt{\lambda_n(L_{q_1})} - \sqrt{\lambda_n(L_{q_2})} \to 0 \text{ as } n \to \infty$$

then their reference Laplacians are isospectral.

For two magnetic Schrödinger operators $L^{st}_{q_1,a_1}$, $L^{st}_{q_2,a_2}$ if

$$\sqrt{\lambda_n(L^{st}_{q_1,a_1})} - \sqrt{\lambda_n(L^{st}_{q_2,a_2})} \to 0 \text{ as } n \to \infty$$

then L^{st}_{0,a_1} is isospectral to L^{st}_{0,a_2}.

Magnetic Ambarzumian Theorem

Theorem. *If* $\lambda_1(L^{st}_{q,a}) \geq 0$ *and* $\lambda_n(L^{st}_{q,a}) - \lambda_n(L^{st}_{0,0}) \to 0$ *then* $q \equiv 0$ *and* a *can be globally eliminated.*

Proof. If $\lambda_n(L^{st}_{q,a}) - \lambda_n(L^{st}_{0,0}) \to 0$ then $L^{st}_{0,a}$ and $L^{st}_{0,0}$ are isospectral.

Then $\lambda_1(L^{st}_{0,a}) = \lambda_1(L^{st}_{0,0}) = 0$ so the magnetic potential can be globally eliminated.

Since a can be eliminated $L^{st}_{q,0}$ is isospectral to $L^{st}_{q,a}$.

Then by the theorem of Davies $q \equiv 0$. ☐

Generalizations

There exists several other Ambarzumian type theorems such as

1. Nicaise (1987): For an operator L^{st}_0 on Γ, $\lambda_2 = (\frac{\pi}{\ell(\Gamma)})^2$ iff Γ is an interval
2. Boman, Kurasov, Suhr (2018): If L^{st}_q on Γ is isospectral to L^{st}_0 on an interval then $q \equiv 0$ and Γ is an interval
3. Bifulco and Kerner (2024): For certain operators[*] L^σ_q with delta coupling coefficients σ: If L^σ_q is isospectral to L^{st}_0 then $q \equiv 0$ and $\sigma = 0$

These results can also be extended to include magnetic potential. For (1) we consider an operator $L^{st}_{0,a}$. If $\lambda_1 = 0$, $\lambda_2 = (\frac{\pi}{\ell(\Gamma)})^2$ then a can be eliminated and Γ is an interval. For (2) we assume $L^{st}_{q,a}$ is isospectral to a Laplacian on an interval. Then a can be eliminated $q \equiv 0$ and Γ is an interval. For (3) we assume ${L^\sigma_{q,a}}^*$ is isospectral to $L^{st}_{0,0}$. Then a can be eliminated $q \equiv 0$ and $\sigma = 0$.

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[*] We require the operator to satisfy any of the 3 following conditions:

$\sum_m \sigma_m \left(1 - \frac{2}{\deg(v^m)}\right) \leq 0$, **or** $\int_\Gamma q dx \geq 0$ and all $\sigma_m \geq 0$,

or $\left(\dfrac{1}{2} \min\limits_m \deg(v^m) - 1\right) \displaystyle\int_\Gamma q dx \geq 0$ and all $\sigma_m \leq 0$.