

FRACTIONAL TORSIONAL RIGIDITY OF COMPACT METRIC GRAPHS

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Introduction

Torsional rigidity is a classical quantity from elasticity theory that measures how resistant a bar is to twisting. For a domain $\Omega \subset \mathbb{R}^d$, the torsional rigidity is defined by

$$T(\Omega) = \int_{\Omega} v(x) dx,$$

where v is the **torsion function**, i.e. the unique solution of the Dirichlet problem

$$\begin{cases} -\Delta v(x) = 1, & x \in \Omega, \\ v(x) = 0, & x \in \partial\Omega. \end{cases}$$

Saint-Venant Inequality

Pólya observed that torsional rigidity is in fact a **geometric quantity** depending only on the shape and size of the domain [3]. In the plane, he proved that among all domains with a fixed area, the **disc maximizes torsional rigidity** [3, 4]. More precisely, for any bounded domain $\Omega \subset \mathbb{R}^2$,

$$T(\Omega) \leq T(O) = \frac{|\Omega|^2}{2\pi},$$

where O is the disc with area $|\Omega|$.

Torsional Rigidity on Metric Graphs

Metric Graphs

- Let $G = (V, E)$ be a finite graph with vertex set V and edge set E .
- Each edge $e \in E$ is identified with an interval $e \cong [0, \ell_e]$, $0 < \ell_e < \infty$.
- The corresponding **metric graph** \mathcal{G} is obtained by connecting the endpoints of these intervals according to the structure of G [1].
- The graph \mathcal{G} is said to be **connected** if for every $x, y \in \mathcal{G}$ there exists a path joining x and y .
- The distance between two points $x, y \in \mathcal{G}$ is $\text{dist}_{\mathcal{G}}(x, y) = \inf\{|P| : P \text{ is a path from } x \text{ to } y\}$.
- The **total length** of the graph is $|\mathcal{G}| = \sum_{e \in E} \ell_e$.

Functions on Metric Graphs

- A function $u : \mathcal{G} \rightarrow \mathbb{R}$ is a collection of edge functions $u = (u_e)_{e \in E}$, $u_e : [0, \ell_e] \rightarrow \mathbb{R}$.
- The graph \mathcal{G} carries a natural measure inherited from the Lebesgue measure on each edge.
- Integration over the graph is defined edgewise: $\int_{\mathcal{G}} u(x) dx = \sum_{e \in E} \int_0^{\ell_e} u_e(x) dx$.
- For $1 \leq p \leq \infty$, we define $L^p(\mathcal{G}) = \{u : \mathcal{G} \rightarrow \mathbb{R} : u_e \in L^p(e), e \in E\}$.
- A function u is **continuous at a vertex** $v \in V$ if: $u_e(v) = u_f(v)$, $e \sim v \sim f$.
- The outward **normal derivative** on an edge e at a vertex v is $\frac{\partial u_e}{\partial n}(v) = \begin{cases} u'_e(v), & v = \ell_e \text{ on } e, \\ -u'_e(v), & v = 0 \text{ on } e. \end{cases}$
- For each vertex v , the set of incident edges is $E_v = \{e \in E : e \sim v\}$.

Torsional Rigidity on Metric Graphs

Let $\mathcal{G} = (V, E)$ be a compact metric graph and let $\emptyset \neq V^D \subset V$ be the set of **Dirichlet vertices**. The **torsion function** of \mathcal{G} with respect to V^D is the function v satisfying

$$\begin{cases} -v''_e = 1, & e \in E, \\ v(v) = 0, & v \in V^D \quad (\text{Dirichlet}), \\ \sum_{e \in E_v} \frac{\partial v_e}{\partial n}(v) = 0, & v \in V \setminus V^D \quad (\text{Kirchhoff}). \end{cases}$$

The **torsional rigidity** is

$$T(\mathcal{G}; V^D) = \|v\|_{L^1(\mathcal{G})} = \int_{\mathcal{G}} v(x) dx.$$

For every compact metric graph \mathcal{G} with finitely many edges [2]:

Upper bound

$$T(\mathcal{G}; V^D) \leq \frac{|\mathcal{G}|^3}{3}, \text{ with equality iff } \mathcal{G} = \text{---}.$$

Lower bound

$$T(\mathcal{G}; V^D) \geq \frac{1}{12} \frac{|\mathcal{G}|^3}{|E|^2}, \text{ with equality iff } \mathcal{G} = \text{---}.$$

Fractional Laplace Operator on \mathcal{G}

Let \mathcal{G} be a connected, compact metric graph, and let $\emptyset \neq V^D \subset V$ be the Dirichlet vertex set.

Standard Laplacian $-\Delta_{\mathcal{G}}$. On each edge $e \in E$, the operator acts as $(-\Delta_{\mathcal{G}}f)_e = -f''_e$, with **Dirichlet conditions** on V^D and **Kirchhoff (Neumann) conditions** on $V \setminus V^D$.

Since \mathcal{G} is compact, the spectrum is discrete: $0 < \lambda_1 \leq \lambda_2 \leq \dots \uparrow \infty$, and the eigenfunctions $\{\varphi_n\}_{n \geq 1}$ form an orthonormal basis of $L^2(\mathcal{G})$. Hence every $f \in L^2(\mathcal{G})$ admits the expansion

$$f = \sum_{n=1}^{\infty} \langle f, \varphi_n \rangle \varphi_n \implies -\Delta_{\mathcal{G}}f = \sum_{n=1}^{\infty} \lambda_n \langle f, \varphi_n \rangle \varphi_n.$$

Fractional Laplacian $(-\Delta_{\mathcal{G}})^{\alpha}$. For $\alpha \in (0, 1)$ we define $(-\Delta_{\mathcal{G}})^{\alpha}$:

$$(-\Delta_{\mathcal{G}})^{\alpha} f := \sum_{n=1}^{\infty} \lambda_n^{\alpha} \langle f, \varphi_n \rangle \varphi_n,$$

$$D((-\Delta_{\mathcal{G}})^{\alpha}) = \left\{ f \in L^2(\mathcal{G}) : \sum_{n=1}^{\infty} \lambda_n^{2\alpha} |\langle f, \varphi_n \rangle|^2 < \infty \right\}.$$

Fractional Torsion Function and Torsional Rigidity

Let $\alpha \in (0, 1)$ and let \mathcal{G} be a compact metric graph.

Fractional torsion function is defined as the unique solution of $(-\Delta_{\mathcal{G}})^{\alpha} u_{\alpha} = 1$ on \mathcal{G} .

Fractional torsional rigidity is defined by $T_{\alpha}(\mathcal{G}) := \int_{\mathcal{G}} u_{\alpha}(x) dx$.

Since the spectrum of $-\Delta_{\mathcal{G}}$ is strictly positive, $(-\Delta_{\mathcal{G}})^{\alpha}$ is invertible:

$$u_{\alpha} = ((-\Delta_{\mathcal{G}})^{\alpha})^{-1} 1 = \sum_{n=1}^{\infty} \frac{\langle 1, \varphi_n \rangle}{\lambda_n^{\alpha}} \varphi_n.$$

Regularity. If $\alpha > \frac{1}{2}$, the series converges uniformly on \mathcal{G} : $u_{\alpha} \in C(\mathcal{G})$, $u_{\alpha}|_{V^D} = 0$.

Positivity. $u_{\alpha}(x) > 0$ for all $x \in \mathcal{G} \setminus V^D$.

Fractional torsional rigidity. By the spectral expansion of u_{α} ,

$$T_{\alpha}(\mathcal{G}) = \sum_{n=1}^{\infty} \frac{|\langle 1, \varphi_n \rangle|^2}{\lambda_n^{\alpha}}.$$

Spectral bounds. $\frac{|\langle 1, \varphi_1 \rangle|^2}{\lambda_1^{\alpha}} \leq T_{\alpha}(\mathcal{G}) \leq \frac{|\mathcal{G}|}{\lambda_1^{\alpha}}$.

For $\alpha \in (0, 1)$ we define

$$H_0^{\alpha}(\mathcal{G}) = \left\{ u \in L^2(\mathcal{G}) : \sum_{n=1}^{\infty} \lambda_n^{\alpha} |\langle u, \varphi_n \rangle|^2 < \infty \right\},$$

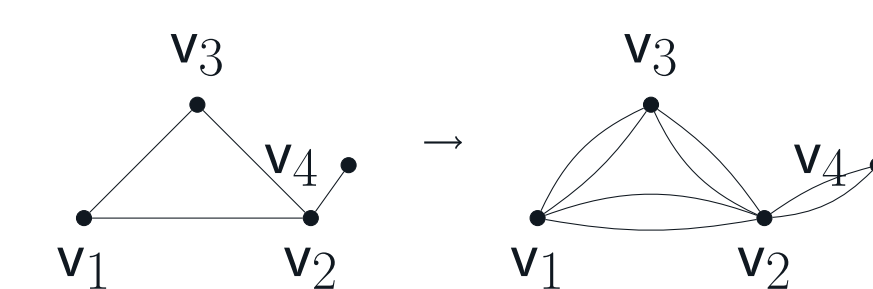
equipped with the norm $\|u\|_{H_0^{\alpha}(\mathcal{G})}^2 = \sum_{n=1}^{\infty} \lambda_n^{\alpha} |\langle u, \varphi_n \rangle|^2 = \langle u, (-\Delta_{\mathcal{G}})^{\alpha} u \rangle$.

Variational characterization of the fractional torsional rigidity is

$$T_{\alpha}(\mathcal{G}) = \sup_{f \in H_0^{\alpha}(\mathcal{G})} \frac{\left(\int_{\mathcal{G}} f(x) dx \right)^2}{\langle f, (-\Delta_{\mathcal{G}})^{\alpha} f \rangle}.$$

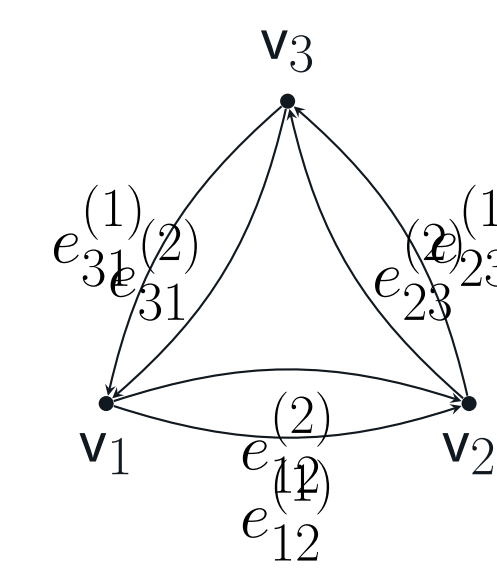
The supremum is attained by a multiple of u_{α} .

Edge doubling.



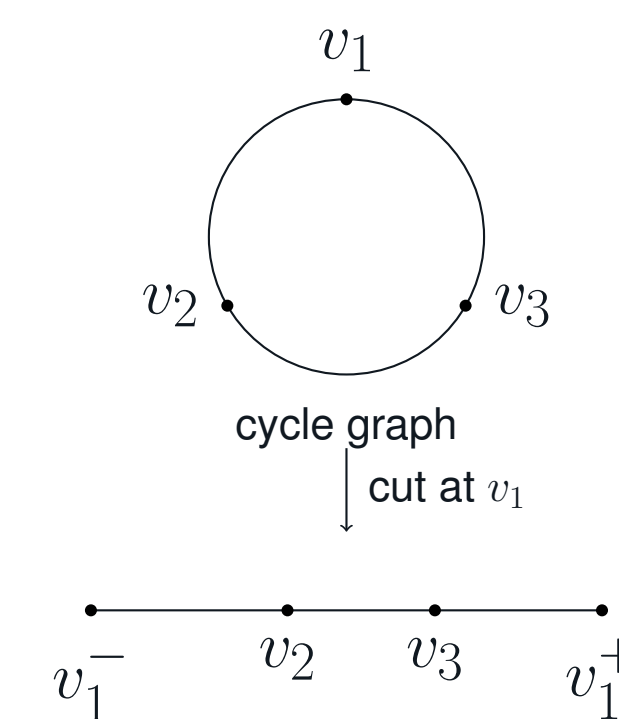
Theorem (Edge Doubling). For $\alpha \in (0, 1)$, if $\tilde{\mathcal{G}}$ is obtained from \mathcal{G} by doubling each edge, then $T_{\alpha}(\tilde{\mathcal{G}}) \leq \frac{1}{2} T_{\alpha}(\mathcal{G})$.

Unfolding to a cycle (even degrees).



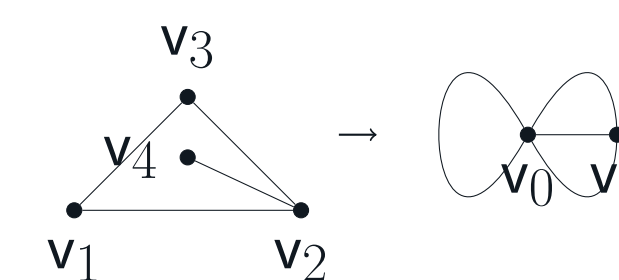
Theorem (Unfolding to a Cycle). If all vertex degrees of \mathcal{G} are even, then there exists a cycle \mathcal{C} with $|\mathcal{C}| = |\mathcal{G}|$ such that $T_{\alpha}(\mathcal{G}) \leq T_{\alpha}(\mathcal{C})$.

Cycle cutting.



Theorem (Cycle Cutting to Interval). If \mathcal{C}_L has a Dirichlet vertex v_0 and \mathcal{I}_L is obtained by cutting at v_0 , then $T_{\alpha}(\mathcal{C}_L) = T_{\alpha}(\mathcal{I}_L)$.

Gluing vertices.



Theorem (Gluing Decreases T_{α}). If \mathcal{G}' is obtained from \mathcal{G} by gluing vertices, then $T_{\alpha}(\mathcal{G}') \leq T_{\alpha}(\mathcal{G})$.

Lower bound (flower). Set $L := |\mathcal{G}|/|E|$ and let \mathcal{F} be the equilateral flower with $|E|$ petals of length L . Then

$$T_{\alpha}(\mathcal{G}) \geq T_{\alpha}(\mathcal{F}) = |E| 8 L^{1+2\alpha} \pi^{-2(1+\alpha)} (1 - 2^{-2(1+\alpha)}) \zeta(2(1+\alpha)).$$

Upper bound (interval). If $\mathcal{I} = [0, |\mathcal{G}|]$ has Dirichlet at one endpoint and Neumann at the other, then

$$T_{\alpha}(\mathcal{G}) \leq T_{\alpha}(\mathcal{I}) = \frac{8^{2\alpha} |\mathcal{G}|^{2\alpha+1}}{\pi^{2(1+\alpha)}} (1 - 2^{-2(1+\alpha)}) \zeta(2(1+\alpha)).$$

References

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